Università di Pisa

Dipartimento di Matematica
Corso di Laurea Triennale in Matematica

The Yamabe Problem

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Anno accademico 2018-2019

Abstract

In this bachelor thesis we introduce and we exhibit the solution of the Yamabe problem for smooth Riemannian manifolds not locally conformally flat of dimension $n \geq 6$. Given a compact Riemannian manifold (M, g) the Yamabe problem consists in finding a metric conformal to g with constant scalar curvature. A significant part of the work is devoted to the set up of a clear and coherent environment where to elegantly solve the problem.

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1 Introduction

The Yamabe problem is a well known problem in Riemannian geometry first proposed by the Japanese mathematician Hidehiko Yamabe in 1960 in reference [Yam60]. The problem is the following:

Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, does there exist a metric conformal to g for which the scalar curvature is a constant?

It was known from the uniformization theorem that the problem had a positive answer for surfaces, which inspired the same question for higher dimensions.

In mathematics *conformal* means angle-preserving and it is a concept that may be adapted to many different structures, whenever the concept of angle is defined. Riemannian manifolds are a very natural environment where to work with angles, as the metric is there precisely to talk about lengths and angles, and therefore our problem arises somehow naturally.

Given a real vector space V and two inner products g and h and considering that an angle between two vectors v, w is defined as

$$\cos\theta = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(w,w)}},$$

it does not require too much thinking to guess that the two products define the same angles if and only if $h = ug$ as bilinear forms for some $u \in \mathbb{R}^+$. By adapting this knowledge to Riemannian manifolds we find that two metrics $g, \tilde{g}$ are conformal if and only if there exists a positive real smooth function u on M such that $\tilde{g} = ug$. To forget the request for the function to be positive and for convenience in calculations this is often written as $\tilde{g} = e^{2f}g$ for some $f \in C^\infty(M)$. The Yamabe problem can then be restated as

Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, does there exist a map $f \in C^\infty(M)$ and a metric $\tilde{g}$ such that $\tilde{g} = e^{2f}g$ and $(M, \tilde{g})$ has constant scalar curvature?

Yamabe claimed in 1960 to have solved the problem with techniques of calculus of variations and partial differential equations, but Neil Trudinger in 1968 discovered a critical error in his proof (see reference [Tru68]). Trudinger managed to repair the proof but only with a restrictive assumption on the manifold M. In 1984, the combined work of Neil Trudinger, Thierry Aubin, and Richard Schoen provided a complete solution to the problem. The problem was answered positively for any compact manifold of dimension greater than three and was solved with techniques from differential geometry, functional analysis, partial differential equations and even general relativity.

In this chapter we set up the environment for the definition of Sobolev spaces on Riemannian manifolds and we recall some results on curvatures which we will need later on. A significant part is devoted to give a rigorous definition of differential operators and to the proofs, in such a rigorous context, of some results concerning them which will also be necessary later on. The theory of differential operators on Riemannian manifolds is quite neat and elegant and, in my opinion, it has not the fame it deserves. Further material on such a topic can be found in reference [Kah80].

In the second chapter we define Sobolev and Hölder spaces on Riemannian manifolds and recall the relations between them. We also recall some results which will turn out necessary to set up the Yamabe problem in a variational environment and prove some of them.

In the third chapter we turn the Yamabe problem into a variational problem and we introduce the Yamabe invariant $\lambda(M)$, which is a geometric invariant defined for a generic compact Riemannian manifold and which is central in the solution of the problem. Moreover, we solve the problem on the sphere (S^n) and show that it always holds that $\lambda(M) \leq \lambda(S^n)$ and that $\lambda(M) < \lambda(S^n)$ is a sufficient condition for the problem to have a solution on M.

Finally, in the fourth chapter, we introduce an interesting tool of conformal differential geometry (first introduced by Parker and Lee in reference [LP87]), that is to say conformal normal coordinates. With this tool we are able to exploit the local geometry of smooth manifolds and prove that the Yamabe problem has a solution in the special case of non locally conformally flat manifolds of dimension equal to or greater than 6.

At the end of the last chapter we briefly discuss the remaining cases, that is to say 3, 4 and 5 dimensional manifolds and locally conformally flat manifolds of dimension $n \geq 6$, for which however, it is not possible to exploit the local geometry and which require for the solution some tools of general relativity. The proof in such cases is therefore beyond the possibilities of this bachelor thesis.

1.1 Preamble definitions

Let (M, g) be a smooth Riemannian manifold of dimension n. In order to set up our problem as a variational problem and define, in the next chapter, what is a Sobolev space on M, we need some preamble definitions.

Let us warm up with a lemma which proves the equivalence of two notations that we will use interchangeably. Here and in the rest of this thesis we will assume tangent vectors to be defined as derivations.

Lemma 1.1. *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on our manifold and let $f \in C^\infty(M)$ be a smooth function. Then*

$$X(f) = df(X).$$

Proof. We are going to prove something which is slightly more general, that is to say that given $p \in M$ and any $v \in T_pM$ it holds

$$v(f) = df(v).$$

The above equality makes sense because by definition of derivation $v(f)$ is a real number and $df : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$, thanks to the canonical identification $T_a\mathbb{R} \to \mathbb{R}$

which is valid for all $a \in \mathbb{R}$ and is given by $v \mapsto v(\mathrm{id})$. Indeed every derivation on $\mathbb{R}$ is written $a\frac{d}{dx}$ for some $a \in \mathbb{R}$ and clearly we want $a\frac{d}{dx} \mapsto a = a\frac{d}{dx}(\mathrm{id})$. By definition of differential

$$df(v)(\mathrm{id}) = v(\mathrm{id} \circ f) = v(f),$$

that is, what we wanted to prove. □

We proceed by recalling how the covariant derivative behaves on tensor fields other than vector fields. If E is a vector bundle on M, we let $\Gamma(E)$ denote its sections.

Proposition 1.2. *If ∇ is a connection on M, there exists only one way to define for all $h, k \in \mathbb{N}$ a new connection on $\mathcal{T}^h_k(M)$, which we will also denote with ∇ such that the following conditions hold:*

1. *On TM the new connection is the same as the given connection.*
2. *On $\mathcal{T}^0 M = C^\infty(M)$ we have $\nabla_X f = X(f)\,(= df(X))$.*
3. *If $K \in \Gamma(\mathcal{T}^h_k(M))$ and $H \in \Gamma(\mathcal{T}^l_m(M))$ and $X \in \mathfrak{X}(M)$ we have*
$$\nabla_X(K \otimes H) = \nabla_X K \otimes H + K \otimes \nabla_X H.$$
4. *∇ commutes with all contractions.*

By leaving the vector field X blank, ∇ becomes a linear operator

$$\nabla : \Gamma(\mathcal{T}^h_k(M)) \to \Gamma(\mathcal{T}^{h+1}_k(M))$$
$$K \mapsto \nabla K$$

where $\nabla K(X, Y_1, ..., Y_h, Y^1, ..., Y^k) = \nabla_X K(Y_1, ..., Y_h, Y^1, ..., Y^k)$. For example, thanks to Lemma 1.1 we have that $\nabla f = df$.

Therefore, if we let ∇ be the Levi-Civita connection on M, it makes sense to consider the kth covariant derivative of a function, which is a k-tensor. Coordinates originating from the differentiation of a tensor $\nabla^m T$ will be separated by a comma. For example, if T is a tensor which in coordinates is written T_{ij}, the coordinates of $\nabla^2 T$ will be written $T_{ij,kl}$.

When integrating on M we will usually integrate with respect to the Riemannian density dV_g, that is the absolute value of the n-form induced by the metric g (in local coordinates $dV_g = \sqrt{\det g}\; dx_1 \wedge ... \wedge dx_n$). In most cases this can be thought just as the volume form derived from the metric but if the manifold is not oriented such a form does not exist on the whole manifold and this is why we need to talk about density. Because of this canonical way of defining a measure on a Riemannian manifold, we can canonically define Lebesgue spaces on it.

Definition 1.1. If $q \in [1, +\infty)$ we let $L^q(M)$ be the set of locally integrable function u for which there exists finite the norm

$$\|u\|_q = \left(\int_M |u|^q dV_g\right)^{\frac{1}{q}}.$$

A key step to define Sobolev spaces on Riemannian manifolds is defining what weak derivatives are. This is done basically the same way as in $\mathbb{R}^n$, with the sole adaptation of having to define the weak derivative for a generic differential operator on M instead of a simpler partial derivative. This is because we do not have a canonical definition of partial derivative on a smooth manifold. Before defining weak derivatives in the context of differential operators we are first going to review such operators.

1.1.1 A short digression on differential operators

To us, a differential operator on the euclidean space is a linear combination of partial derivatives, where coefficients are taken among smooth functions. The *order* of such an operator is just the highest order of the partial derivatives in the linear combination. Because of non-existence of canonical partial derivatives on a smooth manifold, defining what exactly means a differential operator in such a context requires some caution. Because of the important role that differential operators will have in the rest of the thesis we will spend a couple of pages defining them. In this introduction we will follow a general coordinate-free approach.

Definition 1.2. A linear differential operator is a linear map $P : C^\infty(M) \to C^\infty(M)$ such that

$$\forall f \in C^\infty(M) \quad \mathrm{Supp}(P(f)) \subseteq \mathrm{Supp}(f)$$

(recall that $\mathrm{Supp}(f) = \overline{\{x \in M | f(x) \neq 0\}}$).

This definition is simple, elegant and coordinate-independent. But most importantly it is easily generalizable to linear maps $P : \Gamma(E_1) \to \Gamma(E_2)$ where E_1 and E_2 are two generic complex vector bundles on M. Indeed we could define

Definition 1.3. Given an n-dimensional smooth manifold M and two smooth, vector bundles over M, say $E_1 \twoheadrightarrow M$ and $E_2 \twoheadrightarrow M$ let $\Gamma(E_i)$ be the vector space of sections of the bundle. A linear differential operator is a linear map

$$P : \Gamma(E_1) \to \Gamma(E_2)$$

such that

$$\forall s \in \Gamma(E_1) \quad \mathrm{Supp}(P(s)) \subseteq \mathrm{Supp}(s).$$

With such a generalization we can therefore consider general differential operators on vector fields, differential forms, or even tensor fields. For example, things like the exterior derivative, the Lie derivative, the covariant derivative and much more can be interpreted as differential operators. In this thesis we will mainly deal with differential operators on smooth functions and we will therefore state the most important results of the theory only in this particular case. For details on the theory of general linear differential operators on smooth manifolds, see the well-written reference [Kah80].

The only problem that the given definition may have is that it is not clear what derivations have to do with such general linear maps. We will come to this in a moment. By now, however, we can observe that what we intuitively want to be a differential operator satisfies this definition, as we certainly want it to be a linear map, and moreover if a function vanishes in a neighborhood of a point we want its "derivative" to vanish as well, which, switching to complements, is equivalent to require $\mathrm{Supp}(P(f)) \subseteq \mathrm{Supp}(f)$. Moreover, as this proposition shows, differential operators make up a vector space (or more precisely a $C^\infty(M)$-module), which is also something we would intuitively require

Proposition 1.3. *Let P and Q be two differential operator on a manifold M. Then $\forall g, h \in C^\infty(M)$ $gP + hQ$ is a differential operator. In particular, multiplications with functions ($f \mapsto g \cdot f$) are differential operators. Moreover, the composition of differential operators is a differential operator.*

Proof. If $g = 0$ clearly it holds

$$\forall f \in C^\infty(M) \quad \operatorname{Supp}(gPf) = \operatorname{Supp}(0) = \emptyset \subseteq \operatorname{Supp}(f),$$

which also shows that the constant 0 map, that is to say the zero of our vector space, is a differential operator. If, on the other hand, $g \in C^\infty(\mathbb{R}) \setminus \{0\}$ we have that $Pf(p) = 0 \Rightarrow g(p)Pf(p) = 0$ and hence

$$\forall f \in C^\infty(M) \quad \operatorname{Supp}(g \cdot Pf) \subseteq \operatorname{Supp}(Pf) \subseteq \operatorname{Supp}(f).$$

Therefore we only need to prove that the sum of differential operators is a differential operator. This is true because

$$\begin{aligned}\forall f \in C^\infty(M) \quad \operatorname{Supp}((P+Q)f) &= \operatorname{Supp}(Pf + Qf) = \\ &= \operatorname{Supp}(Pf) \cup \operatorname{Supp}(Qf) \subseteq \operatorname{Supp}(f) \cup \operatorname{Supp}(f) = \operatorname{Supp}(f).\end{aligned}$$

The identity operator is a differential operator since it leaves the support unchanged. This implies that multiplications by a function is a differential operator.

Finally

$$\forall f \in C^\infty(M) \quad \operatorname{Supp}(QPf) \subseteq \operatorname{Supp}(Pf) \subseteq \operatorname{Supp}(f),$$

which proves that $Q \circ P$ is a differential operator. □

Once all these properties have convinced us that this abstract definition might work, we need to show the converse, that is to say that a function which satisfies this definition is "intuitively" a differential operator.

Firstly, let us define a central concept in the theory of differential operators.

Definition 1.4. Let P be a differential operator on M. We say that P has *order* m at $x_0 \in M$ if m is the largest nonnegative integer such that there is a smooth function f vanishing at x_0 such that

$$P(f^m)(x_0) \neq 0.$$

We thus write $\operatorname{Ord}(P, x_0) = m$. We then define the order of P as

$$\operatorname{Ord} P = \sup_{x \in M} \{\operatorname{Ord}(P, x)\}.$$

It is not too hard to check that this definition agrees with the usual definition on $\mathbb{R}^n$. As an example consider P on $\mathbb{R}$ defined by

$$P(g) = \frac{d^m g}{dx^m}.$$

If $f(x_0) = 0$, it is immediate that $P(f^r)(x_0) = \frac{d^m f^r}{dx^m}(x_0) = 0$ for every $r > m$. On the other hand if $r = m$ let $f(x) = x$ and observe that

$$\frac{d^m x^m}{dx^m} = m! \neq 0.$$

A problem that arises within this definition is that even in the simplest cases, if M is not compact, the order of a linear differential operator may not exist. For example, if $M = \mathbb{R}$, take for each $n \in \mathbb{N}$ a function

$$\phi_n \in C^\infty(\mathbb{R})$$

such that $\mathrm{Supp}(\phi_n) \subseteq [n, n+1]$ and $\phi_n(n + \frac{1}{2}) > 0$. We then define

$$P(f)(x) = \sum_{n \in \mathbb{N}} \phi_n(x) \frac{d^n f}{dx^n}.$$

It is not hard to see that that for this linear differential operator the order is not defined. In such cases we will say that P is an operator of infinite order.

We will conclude this brief introduction to differential operators by stating the theorem that clarifies why we call such linear operators differential operators. We will first state it on $\mathbb{R}^n$ and then prove it on general Riemannian manifolds. If A, B are two subsets of a topological space X we say that A is compactly contained in B and write $A \Subset B$ if $\overline{A} \subseteq B$ and $\overline{A}$ is compact.

Theorem 1.4 (Local Petree theorem). *Let $\Omega \subseteq \mathbb{R}^n$ and let P be a linear differential operator on Ω (in the sense of the new definition). Let $\Omega_1 \Subset \Omega$. Then there exist an $m \in \mathbb{N}$ such that for every multi-index α such that $|\alpha| \leq m$ there are maps*

$$a_\alpha \in C^\infty(\Omega_1)$$

such that for any $f \in C^\infty(\Omega_1)$

$$(Pf)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha f)(x) \quad \forall x \in \Omega_1.$$

In order to prove such a theorem it is necessary to expand a bit more the theory of differential operators, which, however, is beyond the scope of this thesis (the proof of an even more general statement can be found of course in reference [Kah80]). As anticipated, this theorem translates globally on smooth manifolds

Corollary 1.5 (Global Petree theorem). *Let M be a smooth manifold and let P be a differential operator on M. Let $x_0 \in M$. Then there is a neighborhood $U \ni x_0$, a chart $\varphi : U \to \Omega \subseteq \mathbb{R}^n$ and a positive integer m such that*

$$((Pf) \circ \varphi^{-1})(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha(f \circ \varphi^{-1}))(x) \quad \forall x \in \Omega \; \forall f \in C^\infty(U).$$

Proof. Let $U \ni x_0$ be a neighborhood of x_0 such that there is a chart $\varphi : U \to \Omega \in \mathbb{R}^n$. Define $\tilde{P} : g \in C^\infty(\Omega) \mapsto (P(g \circ \varphi)) \circ \varphi^{-1}$, which is clearly a differential operator, and apply Theorem 1.4 to get

$$\left(\tilde{P}g\right)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha g)(x) \quad \forall x \in \Omega_1 \Subset \Omega.$$

By restricting φ we can suppose $\Omega_1 = \Omega$. We therefore get

$$(Pf)(x) = \tilde{P}(f \circ \varphi^{-1})(\varphi(x)) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha(f \circ \varphi^{-1}))(\varphi(x)) \quad \forall x \in U,$$

which is equivalent to the thesis. $\square$

By locally defining partial derivatives for M as

$$D_\varphi^\alpha f \stackrel{\text{def}}{=} (D^\alpha(f \circ \varphi^{-1})) \circ \varphi,$$

the thesis of Theorem 1.5 may be restated as

$$(Pf)(x) = \sum_{|\alpha| \le m} a_\alpha(x)(D_\varphi^\alpha f)(x) \quad \forall x \in U,$$

which really looks like the definition of a differential operator in $\mathbb{R}^n$.

This theorem shows that a linear differential operator has a meaningful order locally. It is thus a straightforward corollary that the order is well defined for compact manifolds.

The condition on the support proves essential in these theorems. On the other hand there are easy counterexamples proving that if we remove such a condition there are linear operators that locally are anything but euclidean differential operators.

1.1.2 Gradient, divergence and Laplace-Beltrami

An easy example of differential operator of order 1 is the covariant derivative ∇, since it certainly holds that for every real smooth functions $\operatorname{Supp}(\nabla f) \subseteq \operatorname{Supp}(f)$. More generally, the covariant derivative is a differential operator on each tensor field, in the broader sense of Definition 1.3.

We now turn our attention to a particular differential operator, that is to say the Laplace-Beltrami operator. It is a generalization of Laplace's operator on the Euclidean space, and it is extended to general Riemannian manifolds by exploiting the property that Laplace's operator is equal to the divergence of the gradient. Let us therefore generalize these two operators to Riemannian manifolds first. The divergence of a vector field $X \in \mathfrak{X}(M)$ is the scalar function $\operatorname{div} X \in C^\infty(M)$ such that

$$(\operatorname{div} X) dV_g = \mathcal{L}_X dV_g,$$

where $\mathcal{L}_X dV_g$ is the Lie derivative of the n-form dV_g. It is well defined because the n-forms space is one dimensional and therefore it can only change by a scalar. This definition agrees with our intuition of what the divergence is. Indeed, if we are close to a well of a vector field the flow generated by the field will compress the volume around the well and therefore the Lie derivative of the volume form will be negative. A similar situation holds for sources. Alternatively divergence can be defined through the connection as

$$\operatorname{div} X = \operatorname{Tr}(\nabla X).$$

In both definitions it depends on the metric (recall that we are considering the Levi-Civita connection). The divergence is a differential operator of order 1 in the broader definition that generalizes ours and which is mentioned above. However, we will mainly need it to define the Laplace-Beltrami operator, and therefore we will not delve deeper into its properties as a differential operator. We will mainly need one result concerning the divergence:

Lemma 1.6. *The divergence and the exterior derivative (or the covariant derivative, which we saw on functions to be the same thing in Proposition 1.2.2) are formal adjoints. This means that for any compactly supported function $f \in C_C^\infty(M)$ and for any vector field $X \in \mathfrak{X}(M)$ we have that*

$$\int_M df(X)\, dV_g = -\int_M f \operatorname{div}(X)\, dV_g.$$

Proof. By Stokes theorem we get that

$$\begin{aligned}\int_M (f \operatorname{div}(X) + X(f))\, dV_g &= \int_M (f\mathcal{L}_X + \mathcal{L}_X(f))\ dV_g \\ &= \int_M \mathcal{L}_X\,(f dV_g) = \int_M \mathrm{d}\,(\iota_X\,(f dV_g)) = \int_{\partial M} \iota_X\,(f dV_g)\,.\end{aligned}$$

If f has compact support (or if the M has no boundary for non compactly supported functions) the last integral vanishes, so the thesis follows. □

The gradient is basically defined the same way as in the Euclidean space, that is to say, provided that we have a metric, we can use the representation theorem and define the gradient as the vector field that satisfies

$$\langle (\operatorname{grad} f)(x), Y\rangle = df_x(Y) = (Y(f))(x) \quad Y \in T_x M.$$

Also the gradient is a differential operator of order 1 in the broader sense. The following proposition generalizes to Riemannian manifolds a result involving the gradient which is known for $\mathbb{R}^n$.

Proposition 1.7. *Let $f : M \to \mathbb{R}$ be a smooth function, $a < b \in \mathbb{R}$ and $\gamma : [a, b] \to M$ a curve. Then*

$$f(b) - f(a) = \int_a^b \langle \operatorname{grad}(f)(\gamma(t)), \gamma'(t)\rangle dt,$$

where the inner product is of course meant to be that of the metric of the manifold.

Proof. One of the consequences of Stokes theorem is that $f(b) - f(a) = \int_\gamma df$, hence

$$\begin{aligned}f(b) - f(a) &= \int_\gamma df \\ &= \int_a^b df_{\gamma(t)}(\gamma'(t))dt \\ &= \int_a^b \langle \operatorname{grad}(f)(\gamma(t)), \gamma'(t)\rangle\end{aligned}$$

by the very definition of gradient. □

Finally we define the Laplace-Beltrami operator as

$$\Delta f = -\operatorname{div}(\operatorname{grad} f).$$

As neither the divergence nor the gradient are independent from the metric, we do not expect Laplace operator to be either. Indeed, it turns out that it is dependent on the metric. It is an second order differential operator as can be seen by taking a chart in normal coordinates x^i around a point, by considering the locally defined function x^1 and by using the definition of order.

In coordinates we can see from the definition that

$$\Delta f = -(\det g)^{-1/2}\partial_i \left(g^{ij}(\det g)^{1/2}\partial_j f\right), \tag{1.1}$$

or, more easily, we can see it as the contraction of the double covariant derivative

$$\Delta f = -g^{ij} f_{,ij} = -g^{ij}\nabla_i\nabla_j f.$$

The equivalence of the two expressions can be proved by expanding the expression above and by making use of the formulas

$$\Gamma^k_{km} = \frac{1}{\sqrt{\det g}}\partial_m\left(\sqrt{\det g}\right)$$

and

$$\partial_k(g_{jl}) = \Gamma^m_{kj}g_{lm} + \Gamma^k_{lm}g_{kj}.$$

It is important to note that Laplace-Beltrami operator is self-adjoint. In order to see this we first prove the following proposition

Proposition 1.8. *Let $f, h : M \to \mathbb{R}$ be smooth functions. We have that*

$$\int_M f\Delta h\, dV_g = -\int_M \langle df, dh\rangle\, dV_g.$$

Proof. By Lemma 1.6 and by the definition of Laplace-Beltrami operator we get

$$\int_M f\Delta h\, dV_g = \int_M f \operatorname{div}(\operatorname{grad} h)\, dV_g = -\int_M dh(\operatorname{grad} f)\, dV_g = -\int_M \langle df, dh\rangle\, dV_g$$

where the last equality can be seen in coordinates. If one prefers, by definition of gradient it also holds that

$$\int_M dh(\operatorname{grad} f)\, dV_g = -\int_M \langle \operatorname{grad} f, \operatorname{grad} h\rangle\, dV_g$$

and again it is not hard to see in coordinates that the inner product of gradients equals the inner product of differentials. □

The self-adjointedness is now a simple corollary.

Corollary 1.9. *Let $f, h \in C^\infty_C(M)$ be compactly supported functions. We have that*

$$\int_M f\Delta h\, dV_g = \int_M h\Delta f\, dV_g.$$

Proof.

$$\int_M f\Delta h\, dV_g = -\int_M \langle df, dh\rangle\, dV_g = -\int_M \langle dh, df\rangle\, dV_g = \int_M h\Delta f\, dV_g.$$

□

Since, as we just saw, differential operators defined through the metric usually depend on the metric used, when we are dealing with more than one metric, we will denote with a subscript the metric with respect to which the differential operator is meant. For example, ∇_g will denote the covariant derivative with respect to the metric g.

For the development of the thesis it is convenient to derive how the Laplacian transforms under conformal transformation.

Proposition 1.10. *Let (M, g) be a Riemannian manifold and let $\tilde{g} = e^{2f}g$ be a metric conformal to g. The Laplacian transforms accordingly to the formula*

$$\Delta_{\tilde{g}} = e^{-2f}\Delta_g + (n-2)e^{-2f}g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}.$$

Proof. By the expression of the Laplacian in coordinates

$$\begin{aligned}
\Delta_{\tilde{g}} &= \frac{1}{\sqrt{|\det \widetilde{g}|}} \frac{\partial}{\partial x_j} \left(\sqrt{|\det \widetilde{g}|} \tilde{g}^{ij} \frac{\partial}{\partial x^i} \right) \\
&= \frac{1}{e^{nf} \sqrt{|\det g}} \frac{\partial}{\partial x_j} \left(e^{nf} \sqrt{|\det g|} e^{-2f} g^{ij} \frac{\partial}{\partial x^i} \right) \\
&= \frac{1}{e^{nf} \sqrt{|\det g}} \frac{\partial}{\partial x_j} \left(e^{(n-2)f} \sqrt{|\det g|} g^{ij} \frac{\partial}{\partial x^i} \right) \\
&= \frac{1}{e^{2f} \sqrt{|\det g|}} \frac{\partial}{\partial x_j} \left(\sqrt{|\det g|} g^{ij} \frac{\partial}{\partial x^i} \right) + \frac{1}{e^{nf} \sqrt{|\det g|}} \frac{\partial}{\partial x_j} \left(e^{(n-2)f} \right) \sqrt{|\det g|} \frac{\partial}{\partial x^i} \\
&= e^{-2f} \Delta_g + (n-2) e^{-2f} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}
\end{aligned}$$

where we used that $\sqrt{|\det \widetilde{g}|} = e^{n\varphi} \sqrt{|\det g|}$ which can be proved by noting that $\det \widetilde{g} = e^{2n\varphi} \det g$. ☐

1.1.3 Curvatures

We are know going to recall some useful facts about curvatures that we will need later on in the thesis. Our main reference for this section are the notes by Bruno Martelli (reference [Mar19]), which are an intuitive yet concise introduction to differential geometry.

Let (M, g) be a smooth Riemannian manifold. The Riemann curvature tensor is the one which has the complete information about the curvature of the manifold.

Definition 1.5. The *Riemann curvature tensor* R is a tensor field on M of type $(1, 3)$ defined as follows. For every point $p \in M$ and vectors $u, v, w \in T_pM$ we set

$$R(u, v)w = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y](p)} Z$$

where X, Y, Z are vector fields extending u, v, w on some neighbourhood of p and where of course ∇ is the Levi-Civita connection.

The tensor is written in coordinates as follows

$$R(u, v, w)^i = R^i{}_{jkl} u^k v^l w^j$$

(note the unusual order of coordinates). Often we consider instead the $(0, 4)$-tensor

$$R_{ijkl} = R^m{}_{jkl} g_{im}.$$

An interesting feature of the Riemann curvature tensor, which also contributes in giving it a "physical" meaning, is that it contains all the second order information on how the metric tensor changes on a small neighborhood in normal coordinates.

Proposition 1.11. *In normal coordinates we have*

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(0) x^k x^l + O\left(|x|^3\right).$$

It is also handy to recall the symmetries of the Riemann tensor, which are always necessary in calculations with coordinates.

Proposition 1.12. *The following symmetries hold in any coordinate chart:*

- $R_{ijkl} = -R_{jikl} = -R_{ijlk}$
- $R_{ijkl} = R_{klij}$
- $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ *(known as the* ***first Bianchi identity****)*
- $R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$ *(known as the* ***second Bianchi identity****)*

Moreover, for any one form ω we have that

$$\omega_{j,kl} - \omega_{j,lk} = R^i{}_{jkl}\omega_i.$$

Being the Riemann tensor of type $(1,3)$ it is natural to study its contractions. There are three possible contractions:

$$R^k{}_{kij} \qquad R^k{}_{ikj} \qquad R^k{}_{ijk}.$$

By the symmetries above the first one vanishes and the second and the third ones differ by a sign, so there is essentially only one way to get a non trivial tensor by contraction, which results in what is called the *Ricci tensor*, defined as

$$R_{ij} = R^k{}_{ikj}.$$

Proposition 1.13. *The Ricci tensor is symmetric.*

Proof. By the symmetries of the Riemann tensor

$$R_{ij} = R^k{}_{ikj} = R_{hikj}g^{hk} = R_{kjhi}g^{hk} = R^h{}_{jhi} = R_{ji}.$$

□

Even if it is symmetric, unlike the metric tensor, the Ricci tensor needs not to be positive definite and it can also be degenerate. For example on $\mathbb{R}^n$ with the standard metric, the Riemann tensor vanishes and therefore also the Ricci tensor does.

The geometric information contained in the Ricci tensor is that of the change of the determinant of the metric tensor in normal coordinates (so less information than in the Riemann tensor, but in the same spirit).

Proposition 1.14. *In normal coordinates we have*

$$\det g_{ij}(x) = 1 - \frac{1}{3}R_{ij}(0)x^i x^j + O\left(|x|^3\right).$$

Proof. For any A real or complex $n \times n$ matrix we have

$$\det(\mathbb{1} + A) = 1 + \mathrm{Tr}(A) + O\left(\|A\|^2\right).$$

By applying this formula to the result of Proposition 1.11 we have

$$\det g_{ij}(x) = 1 - \frac{1}{3}R^i_{kil}(0)x^k x' + O\left(|x|^3\right) = 1 - \frac{1}{3}R_{kl}(0)x^k x' + O\left(|x|^3\right),$$

which completes the proof. □

From this we get an important corollary about volume forms, which will turn out handy in the calculations.

Corollary 1.15. *Let dV_g be the Riemannian density on M. In normal coordinates we have*

$$dV_g = \left(1 - \frac{1}{6}R_{ij}(0)x^i x^j + O\left(|x|^3\right)\right) dx^1 \wedge ... \wedge dx^n.$$

Proof. By definition

$$dV_g = \sqrt{\det g_{ij}}\ dx_1 \wedge ... \wedge dx_n.$$

From the previous proposition, by recalling that $\sqrt{1+t} = 1 + \frac{1}{2}t + O(|t|^2)$, we have the thesis. □

Because in the rest of the thesis we are going to deal with conformal transformations pretty often, it is important to derive how the Ricci tensor transforms under such transformations. It is not convenient to derive a formula where the Riemann tensor is expressed in terms of the metric tensor. Instead, we can make use of the expression of the Riemann tensor in terms of the Christoffel symbols, which can be easily computed in terms of the metric tensor.

Proposition 1.16. *Christoffel symbols are expressed in terms of the metric tensor as follows*

$$\Gamma^i{}_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Corollary 1.17. *Let (M, g) be a Riemannian manifold and let $\tilde{g} = e^{2f}g$ be a metric conformal to g. The Christoffel symbols transform accordingly to the formula*

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \left(\delta^k_i \frac{\partial f}{\partial x^j} + \delta^k_j \frac{\partial f}{\partial x^i} - g_{ij}g^{kl}\frac{\partial f}{\partial x^l}\right).$$

Proof. By the above proposition and recalling that g^{kl} is the inverse matrix of g_{kl}, by simple calculations, we have that

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &= \frac{1}{2}\widetilde{g}^{kl}\left(\widetilde{g}_{il,j} + \widetilde{g}_{jl,i} - \widetilde{g}_{ij,l}\right) \\
&= \frac{1}{2}e^{-2f}g^{kl}\left(\frac{\partial}{\partial x^j}\left(e^{2f}g_{il}\right) + \frac{\partial}{\partial x^i}\left(e^{2f}g_{jl}\right)g_{jl,i} - \frac{\partial}{\partial x^l}\left(e^{2f}g_{ij}\right)\right) \\
&= \frac{1}{2}e^{-2f}g^{kl}\left(2e^{2f}\frac{\partial f}{\partial x^j}g_{il} + e^{2f}g_{il,j} + 2e^{2f}\frac{\partial f}{\partial x^i}g_{jl} + e^{2f}g_{jl,i} - 2e^{2f}\frac{\partial f}{\partial x^l}g_{ij} - e^{2f}g_{ij,l}\right) \\
&= \Gamma^k_{ij} + g^{kl}\left(\frac{\partial f}{\partial x^j}g_{il} + \frac{\partial f}{\partial x^i}g_{jl} - \frac{\partial f}{\partial x^l}g_{ij}\right) \\
&= \Gamma^k_{ij} + \left(\delta^k_i\frac{\partial f}{\partial x^j} + \delta^k_j\frac{\partial f}{\partial x^i} - g_{ij}g^{kl}\frac{\partial f}{\partial x^l}\right)
\end{aligned}$$

where the last equality also follows from the fact that g^{kl} is the inverse of g_{kl}. □

In order to derive how the Ricci tensor transforms under conformal transformations we now need to express the Riemann tensor in terms of the Christoffel symbols and then contract it

Lemma 1.18. *The Riemann tensor is expressed in terms of the Christoffel symbols as follows*

$$R^i{}_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x_k} - \frac{\partial \Gamma^i_{kj}}{\partial x_l} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}.$$

Corollary 1.19. *The Ricci tensor is expressed in terms of the Christoffel symbols as follows*

$$R_{jl} = R^k{}_{jkl} = \frac{\partial \Gamma^k_{lj}}{\partial x_k} - \frac{\partial \Gamma^k_{kj}}{\partial x_l} + \Gamma^k_{km}\Gamma^m_{lj} - \Gamma^k_{lm}\Gamma^m_{kj}.$$

Proposition 1.20. *Let (M, g) be a Riemannian manifold and let $\tilde{g} = e^{2f}g$ be a metric conformal to g. The relation between the coordinates of the Ricci tensors of g and $\tilde{g}$ is the following*

$$\tilde{R}_{jk} = R_{jk} - (n-2)f_{,jk} + (n-2)f_j f_k + \left(\Delta_g f - (n-2)f_i f^i\right) g_{jk}, \tag{1.2}$$

where $f_j = \frac{\partial f}{\partial x_j}$ is meant to be the partial derivative (and not the index of the covariant derivative, which we would write $f_{,j}$)

Proof. By the previous corollary we have

$$\begin{aligned}
\widetilde{R}_{jl} &= \frac{\partial \widetilde{\Gamma}^k_{lj}}{\partial x_k} - \frac{\partial \widetilde{\Gamma}^k_{kj}}{\partial x_l} + \widetilde{\Gamma}^k_{km}\widetilde{\Gamma}^m_{lj} - \widetilde{\Gamma}^k_{lm}\widetilde{\Gamma}^m_{kj} \\
&= \frac{\partial}{\partial x_k}\left(\Gamma^k_{lj} + \left(\delta^k_l \frac{\partial f}{\partial x^j} + \delta^k_j \frac{\partial f}{\partial x^l} - g_{lj}g^{ki}\frac{\partial f}{\partial x^i}\right)\right) \\
&- \frac{\partial}{\partial x_l}\left(\Gamma^k_{kj} + \left(\delta^k_k \frac{\partial f}{\partial x^j} + \delta^k_j \frac{\partial f}{\partial x^k} - g_{kj}g^{ki}\frac{\partial f}{\partial x^i}\right)\right) \\
&+ \left(\Gamma^k_{km} + \left(\delta^k_k \frac{\partial f}{\partial x^m} + \delta^k_m \frac{\partial f}{\partial x^k} - g_{km}g^{ki}\frac{\partial f}{\partial x^i}\right)\right)\left(\Gamma^m_{lj} + \left(\delta^m_l \frac{\partial f}{\partial x^j} + \delta^m_j \frac{\partial f}{\partial x^l} - g_{lj}g^{mi}\frac{\partial f}{\partial x^i}\right)\right) \\
&- \left(\Gamma^k_{lm} + \left(\delta^k_l \frac{\partial f}{\partial x^m} + \delta^k_m \frac{\partial f}{\partial x^l} - g_{lm}g^{ki}\frac{\partial f}{\partial x^i}\right)\right)\left(\Gamma^m_{kj} + \left(\delta^m_k \frac{\partial f}{\partial x^j} + \delta^m_j \frac{\partial f}{\partial x^k} - g_{kj}g^{mi}\frac{\partial f}{\partial x^i}\right)\right) \\
&= R_{jl} + \frac{\partial}{\partial x_k}\left(\delta^k_l f_j + \delta^k_j f_l - g_{lj}g^{ki}f_i\right) - \frac{\partial}{\partial x_l}\left(\delta^k_k f_j + \delta^k_j f_k - \delta^i_j f_i\right) \\
&+ \Gamma^m_{lj}\left(\delta^k_k f_m + \delta^k_m f_k - \delta^i_m f_i\right) + \Gamma^k_{km}\left(\delta^m_l f_j + \delta^m_j f_l - g_{lj}g^{mi}f_i\right) \\
&- \Gamma^m_{kj}\left(\delta^k_l f_m + \delta^k_m f_l - g_{lm}g^{ki}f_i\right) - \Gamma^k_{lm}\left(\delta^m_k f_j + \delta^m_j f_k - g_{kj}g^{mi}f_i\right) \\
&+ \left(\delta^k_k f_m + \delta^k_m f_k - \delta^i_m f_i\right)\left(\delta^m_l f_j + \delta^m_j f_l - g_{lj}g^{mi}f_i\right) \\
&- \left(\delta^k_l f_m + \delta^k_m f_l - g_{lm}g^{ki}f_i\right)\left(\delta^m_k f_j + \delta^m_j f_k - g_{kj}g^{mi}f_i\right) \\
&= R_{jl} + f_{jl} + f_{lj} - \frac{\partial}{\partial x_k}\left(g_{lj}g^{ki}f_i\right) - nf_{jl} - f_{jl} + f_{jl} + n\Gamma^m_{lj}f_m + \Gamma^k_{lj}f_k - \Gamma^m_{lj}f_m \\
&+ \Gamma^k_{kl}f_j + \Gamma^k_{kj}f_l - \Gamma^k_{km}g_{lj}g^{mi}f_i - \Gamma^m_{lj}f_m - \Gamma^k_{kj}f_l + \Gamma^m_{kj}g_{lm}g^{ki}f_i - \Gamma^k_{lk}f_j - \Gamma^k_{lj}f_k \\
&+ \Gamma^k_{lm}g_{kj}g^{mi}f_i + (nf_m + f_m - f_m)\left(\delta^m_l f_j + \delta^m_j f_l - g_{lj}g^{mi}f_i\right) \\
&- \left((n+3)f_l f_j - g_{lj}g^{mi}f_i f_m - g_{mj}g^{mi}f_l f_i - g_{lm}g^{mi}f_i f_j - g_{lj}g^{ki}f_i f_k + g_{lm}g^{ki}g_{kj}g^{mu}f_u f_i\right) \\
&= R_{jl} - (n-2)\left(f_{jl} - \Gamma^m_{lj}f_m\right) + (n-2)f_j f_l - (n-2)g_{jl}f_k f^k \\
&- g_{jl}\left(\partial_k\left(g^{ki}f_i\right) + \Gamma^k_{km}g^{mi}f_i\right) - \left(g^{ki}f_i\right)\partial_k(g_{jl}) + \Gamma^m_{kj}g_{lm}g^{ki}f_i + \Gamma^k_{lm}g_{kj}g^{mi}f_i
\end{aligned}$$

$$= R_{jl} - (n-2)f_{,jl} + (n-2)f_jf_l + \left(\Delta_g f - (n-2)f_kf^k\right) g_{jl}$$

where in the last equality we used that

$$f_{,jl} = f_{jl} - \Gamma^m_{lj} f_m,$$

which is a well known identity. Moreover, we used that

$$\Delta_g f = -\partial_k \left(g^{ki} f_i\right) - \Gamma^k_{km} g^{mi} f_i,$$

which can be proved from the coordinates form of the Laplacian. Indeed

$$\Delta_g f = -\frac{1}{\sqrt{\det g}}\partial_m \left(\sqrt{\det g}\, g^{mi}\partial_i f\right) = -\partial_k \left(g^{ki}\partial_i f\right) - \frac{1}{\sqrt{\det g}}\partial_m \left(\sqrt{\det g}\right) g^{mi}\partial_i f,$$

which, combined with

$$\Gamma^k_{km} = \frac{1}{\sqrt{\det g}}\partial_m \left(\sqrt{\det g}\right)$$

proves the wanted identity. Finally we also used that

$$\partial_k(g_{jl}) = \Gamma^m_{kj} g_{lm} + \Gamma^k_{lm} g_{kj},$$

which can be proved simply by substituting in the Christoffel symbols in terms of the metric tensor first derivatives (and which is in general a good way of expressing the first derivatives of the metric tensor). □

The scalar curvature is a smooth real function S on M that is obtained by contracting the Ricci tensor, that is to say

$$S = g^{ij} R_{ij}.$$

In other words, it is the trace of the Ricci tensor. In order to understand what geometric information the scalar curvature brings us, let $p \in M$ be a point and $B_r(p)$ be a geodesic ball of radius r centered at p, which is well defined if r is sufficiently small. We know that the volume of a Euclidian ball $B_r(0) \subseteq \mathbb{R}^n$ is

$$\mathrm{Vol}(B_r(0)) = V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n,$$

where, of course, Γ is the Euler Gamma function.

Proposition 1.21. *We have that*

$$\mathrm{Vol}\left(B_r(p)\right) = V_n(r) \left(1 - \frac{S(p)}{6(n-2)} r^2 + O(r^4)\right).$$

Therefore, the scalar curvature measures at second order the ratio between the volumes of small geodesic balls and Euclidean balls with the same small radius. In particular, if $R(p)$ is negative (respectively, positive), geodesic balls of small radius r centered at p have larger (respectively, smaller) volume than the Euclidean ones with the same radius r.

We also need to know, for future calculations, how the scalar curvature transforms under conformal transformations.

Proposition 1.22. *Let $S : M \to \mathbb{R}$ be the scalar curvature relative to the metric g and $\tilde{S} : M \to \mathbb{R}$ the scalar curvature relative to the metric $\tilde{g} = e^{2f} g$. Then*

$$\tilde{S} = e^{-2f} \left(S + 2(n-1)\Delta_g f - (n-1)(n-2) f_i f^i \right). \tag{1.3}$$

Proof. By definition and by Proposition 1.20 we have

$$\begin{aligned} \tilde{S} = \tilde{g}^{jk} \tilde{R}_{jk} &= e^{-2f} g^{jk} \left(R_{jk} - (n-2) f_{,jk} + (n-2) f_j f_k + \left(\Delta_g f - (n-2) f_i f^i \right) g_{jk} \right) \\ &= e^{-2f} \Big(S + (n-2)\Delta_g f + (n-2) f^k f_k + n\Delta_g f - n(n-2) f_i f^i \Big) \\ &= e^{-2f} \left(S + 2(n-1)\Delta_g f - (n-1)(n-2) f_i f^i \right) \end{aligned}$$

where we used that the Laplacian is the contraction of the double covariant derivative. ☐

Another useful fact about scalar curvature is the following:

Proposition 1.23. *The following relation holds between the differentiated scalar curvature and Ricci tensor*

$$S_{,m} - 2R^i{}_{m,i} = 0.$$

Proof. The second Bianchi identity tells us that

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0.$$

If we contract it on the indices i, k and on j, l, we get

$$g^{jl} g^{ik} R_{ijkl,m} + g^{jl} g^{ik} R_{ijlm,k} + g^{jl} g^{ik} R_{ijmk,l} = 0.$$

The thesis follows from the fact that the covariant derivative commutes with contractions, therefore proving that

$$g^{jl} g^{ik} R_{ijkl,m} = g^{jl} R^k{}_{jkl,m} = g^{jl} R_{jl,m} = S_{,m},$$

$$g^{jl} g^{ik} R_{ijlm,k} = -g^{ik} g^{jl} R_{jilm,k} = -g^{ik} R_{im,k} = -R^i{}_{m,i}$$

and similarly

$$g^{jl} g^{ik} R_{ijmk,l} = -g^{jl} g^{ik} R_{ijkm,l} = -R^i{}_{m,i}.$$

☐

In the development of the solution of the Yamabe problem we will need to make use of the concept of Einstein manifold.

Definition 1.6. In the context of Riemannian geometry, an Einstein manifold is a Riemannian manifold whose Ricci tensor is proportional to the metric. In formula

$$R_{ij} = k \cdot g_{ij}$$

where $k \in \mathbb{R}$ is a constant and g is the metric. A metric which respects such a condition is said to be an Einstein metric. Equivalently a manifold is Einstein if the *traceless Ricci tensor* $B_{ij} = R_{ij} - (S/n) g_{ij}$ vanishes everywhere.

A trivial example of Einstein manifold is the Euclidean space, for which $k = 0$. A less trivial example which is more in our interest is the n-sphere with the standard metric, for which a simple calculation of the Ricci tensor shows that $k = n - 1$.
It follows from Proposition 1.23 that any Einstein manifold has constant scalar curvature.

Finally, we are going to introduce the Weyl tensor (defined for $n \geq 3$):

$$\begin{aligned} W_{iklm} = R_{iklm} &+ \frac{1}{n-2}\left(R_{im}g_{kl} - R_{il}g_{km} + R_{kl}g_{im} - R_{km}g_{il}\right) \\ &+ \frac{1}{(n-1)(n-2)} R\left(g_{il}g_{km} - g_{im}g_{kl}\right). \end{aligned}$$

The formula is chosen so that the trace of W with respect to any two indices vanishes. W vanishes completely if $n = 3$, so it carries useful information only for $n \geq 4$. By definition of W and of B (in Definition 1.6) we can write the Riemann tensor as a sum of three parts.

$$R_{ijki} = W_{ijkl} + \frac{1}{n-2}\left(B_{ik}g_{jl} - B_{il}g_{jk} + B_{jl}g_{ik} - B_{jk}g_{il}\right) + \frac{S}{n(n-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$$

In this sense, the Weyl tensor completes the information provided by the Ricci tensor and the scalar curvature. In particular, if $W = 0$ and $B = 0$, the curvature tensor is completely determined by the scalar curvature S, which is constant by Proposition 1.23, and therefore M is known to have constant curvature.

It is a quite famous result that a complete, simply connected manifold of constant curvature is isometric to $\mathbb{R}^n$, S^n, or to the n-dimensional hyperbolic space.

2 Sobolev spaces on Riemannian Manifolds

Our main goal in this chapter is to introduce some results concerning Sobolev spaces on Riemannian manifolds which will play an essential role in the solution of the Yamabe problem. Our main references for this part are references [Aub13] and [LP87]. The author of the first reference, Aubin, gave a great contribution to the solution of our problem.

In this chapter M is meant to be a generic Riemannian manifold.

2.1 Sobolev and Hölder spaces

We are ready to extend differential operators to broader sets of functions, by defining weak derivatives. Let $P : C^\infty(M) \to C^\infty(M)$ be a linear differential operator on M and let $u, f \in L^1_{\text{loc}}(M)$ (that is the set of locally integrable functions on M). We say that $Pu = f$ in a weak sense, or that $Pu \overset{\text{w}}{=} f$, or equivalently that u is a weak solution to the equation $Pu = f$ if it holds

$$\forall \varphi \in C^\infty_C(M) \quad \int_M uP^*\varphi dV_g = \int_M f\varphi dV_g,$$

where $C^\infty_C(M)$ is the set of smooth compactly supported functions on M and P^* is the adjoint operator of P.

As it happens for the euclidean space, if it holds $Pu \overset{\text{w}}{=} f$ and $Pu \overset{\text{w}}{=} g$, then $f = g$ almost everywhere on M. Equivalently said, if we consider functions, as it is often done, as classes of equivalence through the relation that identifies two functions equal almost everywhere, the "weak derivative" is unique. This weak definition therefore extends differential operators from the smooth functions to larger subsets of $L^1_{\text{loc}}(M)$ (but not on the whole $L^1_{\text{loc}}(M)$: variations of the counterexamples that work in the Euclidian space work here as well). The spaces we are about to define address the necessity of having spaces closed under weak derivations.

Definition 2.1. We define $L^q_k(M)$ as the set of all function $u \in L^q_k(M)$ such that $Pu \in L^q(M)$ (in a weak sense) for all differential operators P of order at most k. We then equip $L^q_k(M)$ with the Sobolev norm $\|\cdot\|_{q,k}$ defined by

$$\|u\|_{q,k} = \left(\sum_{i=0}^{k} \int_M |\nabla^i u|^q dV_g \right)^{\frac{1}{q}}$$

and we call the resulting Banach space a *Sobolev space* on M.

It is important to note that if M is complete $C_C^\infty(M)$ is dense in $L_k^q(M)$.

Similarly, for the space of continuous and k times differentiable bounded functions $C_B^k(M)$ we require not only its functions to be k times continuous and differentiable, but also the norm

$$\|u\|_{C_B^k} = \sum_{i=0}^{k} \sup_M \left|\nabla^i u\right| \tag{2.1}$$

to be finite $\forall u \in C_B^k(M)$.

Finally we define the *Hölder space* $C^{k,\alpha}(M)$ (morally we should write $C_B^{k,\alpha}(M)$, since the conditions below requires them to be bounded, but it makes notation heavy) for $\alpha \in (0,1)$ as the set of $u \in C_B^k(M)$ such that the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C_B^k} + \sup_{x \neq y} \frac{\left|\nabla^k u(x) - \nabla^k u(y)\right|}{|x-y|^\alpha}$$

is finite. The last superior limit is taken over all y contained in a normal coordinate neighborhood of x and $\nabla^k u(y)$ is taken to mean the tensor at x obtained by parallel transport along the radial geodesic from x to y.

Note that if the manifold M is compact as in the case of the Yamabe problem, $|\nabla^k u|$ is bounded for any k for which the kth covariant derivative exists (as it it continuous on a compact set). Therefore, in such cases C_B^k and C^k are in fact the same space. Hence, in the rest of this thesis we will write simply C^k.

We are interested to see how these spaces are embedded into one another. As it is often the case it will be useful explore the Euclidian case first. We have two embedding theorems for Sobolev spaces in $\mathbb{R}^n$.

Theorem 2.1 (Sobolev Embedding theorem in $\mathbb{R}^n$)**.** *Let*

$$\frac{1}{r} = \frac{1}{q} - \frac{k}{n}.$$

Then the inclusion $L_k^q(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ is a continuous embedding. In particular, for $q = 2, k = 1, r = p = \frac{2n}{n-2}$ we have

$$\forall \phi \in L_1^2(\mathbb{R}^n) \quad \|\phi\|_p^2 \leq \sigma_n \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \tag{2.2}$$

for a suitable constant σ_n. This inequality is called the ***Sobolev inequality****. The smallest constant for which inequality 2.2 holds is called n-dimensional Sobolev constant.*

We have another important embedding.

Theorem 2.2. *Let $\alpha \in (0,1)$ and $r \in \mathbb{N}$ and suppose*

$$\frac{1}{q} \leq \frac{k-r-\alpha}{n}.$$

Then $L_k^q(\mathbb{R}^n) \subset C^{r,\alpha}(\mathbb{R}^n)$ and the inclusion $L_k^q(\mathbb{R}^n) \hookrightarrow C^{r,\alpha}(\mathbb{R}^n)$ is a continuous embedding.

As for the proof of such theorems, we refer to specialized texts on Sobolev spaces, as we are only interested on their counterparts on Riemannian manifolds.

The following generalizations of the previous theorems are necessary to set up the variational formulation of the Yamabe problem.

Theorem 2.3. *Let M be a compact Riemannian manifold of dimension n. Suppose*

$$\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n},$$

then $L^q_k(M) \subset L^r(M)$ and the inclusion $L^q_k(M) \hookrightarrow L^r(M)$ is a continuous embedding. If strict inequality holds the inclusion is a compact operator.

Partial proof. We are going to prove $L^q_k(M) \subset L^r(M)$. Let $r' \in \mathbb{R}^+$ be such that

$$\frac{1}{r'} = \frac{1}{q} - \frac{k}{n}.$$

By hypothesis $\frac{1}{r} \geq \frac{1}{r'}$ and therefore $r' \geq r$ and thus $L^{r'} \subset L^r$, since M is compact and therefore is a finite measure space.

Consider for any $P \in M$ a normal coordinates chart around it. Since M is compact we can choose a finite number of them which covers it. Let $\{U_i\}$ be such an open covering. Consider $\{\rho_j\}$ a partition of unity subordinated to $\{U_i\}$. Let $\varphi \in L^q_k(M)$. Then $\varphi \cdot \rho_j \in L^q_k(M)$ by Hölder inequality and by using the fact that ρ_j has compact support (as its support is a closed subset of a compact set). We can now see it as a function in $\mathbb{R}^n$ with compact support, through the normal coordinates chart. Now by making use Corollary 1.15 and by appropriately refining our open cover (although keeping it finite) we can prove that there exists $C \in \mathbb{R}^+$ such that $\|\varphi \cdot \rho_i\|_{L^q_j(M)} \leq (1 + C) \|\varphi \cdot \rho_i\|_{L^q_j(\mathbb{R}^n)}$ thus proving that $\varphi \cdot \rho_i \in L^q_k(\mathbb{R}^n)$ with respect to the Lebesgue measure. By the Sobolev embedding theorem in $\mathbb{R}^n$ we have that $\varphi \cdot \rho_i \in L^{r'}(\mathbb{R}^n)$ and taking it back through the chart we get $\varphi \cdot \rho_i \in L^{r'}(M)$. Because the sum of functions in $L^{r'}(M)$ is in $L^{r'}(M)$, summing over the whole partition of unity, which is finite by the very definition of subordinate partition, we have that $\varphi \in L^{r'}(M) \subset L^r(M)$. □

We also have a Riemannian counterpart of Theorem 2.2. Similarly to the previous theorem we can prove the following

Theorem 2.4. *Let M be a compact Riemannian manifold of dimension n. Let $\alpha \in (0, 1), r \in N$ and*

$$\frac{1}{q} \leq \frac{k - \alpha}{n}.$$

Then $L^q_k(\mathbb{R}^n) \subset C^{r,\alpha}(M)$ and the inclusion $L^q_k(M) \hookrightarrow C^{r,\alpha}(M)$ is a continuous embedding.

We now turn our attention to the inequality 2.2 and look for a similar inequality on Riemannian manifolds. Surprisingly the constant σ_n that we got for $\mathbb{R}^n$ almost works for compact Riemannian manifolds. Indeed the following result, due to Aubin, holds.

Theorem 2.5. *Let (M, g) be a compact Riemannian manifold of dimension n. Let $p = \frac{2n}{n-2}$ and let σ_n be the constant defined in Theorem 2.1. Then for every $\varepsilon \in \mathbb{R}^+$ there exists a constant C_ε such that*

$$\forall \varphi \in C^\infty(M) \quad \|\varphi\|_p^2 \leq (1 + \varepsilon)\sigma_n \int_M |\nabla \varphi|^2 \, dV_g + C_\varepsilon \int_M \varphi^2 dV_g.$$

Proof. Fix $\varepsilon > 0$. Around each point $x \in M$ we can choose a neighborhood U such that, in normal coordinates on U, the eigenvalues of the metric tensor are between $(1 \;+\varepsilon)^{-1}$ and $(1 \;+\varepsilon)$, and furthermore $dV_g = f dx$ where $(1+\varepsilon)^{-1} < f < (1+\varepsilon)$. Choose a finite subcover $\{U_i\}$ and a subordinate partition of unity, which we may write as $\{\alpha_i^2\}$, where $\alpha_i \in C^\infty(M)$ and $\sum_i \alpha_i^2 = 1$. Then we have

$$\begin{aligned}\|\varphi\|_p^2 = \left\|\varphi^2\right\|_{p/2} = \left\|\sum_i \alpha_i^2\varphi^2\right\|_{p/2} &\leqslant \sum_i \left(\int_M |\alpha_i\varphi|^p \, dV_g\right)^{2/p} \\ &\leqslant (1+\varepsilon)^{2/p} \sum_i \left(\int_{U_i} |\alpha_i\varphi|^p \, dx\right)^{2/p}.\end{aligned}$$

The Sobolev inequality on $\mathbb{R}^n$ together with our estimates on the deviation of g and dV_g from the euclidean metric, implies

$$\begin{aligned}\left(\int_{U_i} |\alpha_i\varphi|^p \, dx\right)^{2/p} &\leqslant \sigma_n \int_{U_i} |\nabla(\alpha_i\varphi)|_0^2 \, dx \\ &\leqslant (1+\varepsilon)^2 \sigma_n \int_{U_i} |\nabla(\alpha_i\varphi)|^2 \, dV_g,\end{aligned}$$

where $|\cdot|_0$ represents the Euclidean metric in normal coordinates. Furthermore,

$$\begin{aligned}|\nabla(\alpha_i\varphi)|^2 &= \alpha_i^2|\nabla\varphi|^2 + 2\alpha_i\varphi\,\langle\nabla\alpha_i, \nabla\varphi\rangle + \varphi^2\,|\nabla\alpha_i|^2 \\ &\leqslant (1+\varepsilon)\alpha_i^2|\nabla\varphi|^2 + \left(1+\varepsilon^{-1}\right)\varphi^2\,|\nabla\alpha_i|^2,\end{aligned}$$

where the last line follows from the Cauchy-Schwartz inequality and the inequality $2ab < \varepsilon a^2 + \varepsilon^{-1}b^2$. Thus for small ε it holds

$$\begin{aligned}\|\varphi\|_p^2 &\leqslant (1+4\varepsilon)\sigma_n \sum_i \int_{U_i} \alpha_i^2|\nabla\varphi|^2 dV_g + C_\varepsilon \sum_i \int_{U_i} \varphi^2\,|\nabla\alpha_i|^2\, dV_g \\ &\leqslant (1+4\varepsilon)\sigma_n \int_M |\nabla\varphi|^2 dV_g + C'_\varepsilon \int_M \varphi^2 dV_g.\end{aligned}$$

□

2.2 The Laplacian and Schrödinger's equation

Laplace operator will be central in solving the Yamabe problem, therefore we will state the properties that we will need, generalizing again from the Euclidean case.

Theorem 2.6 (Local elliptic regularity). *Suppose Ω is an open set in $\mathbb{R}^n$, Δ is the Laplacian with respect to any metric on Ω, and $u \in L^1_{loc}(\Omega)$ is a weak solution to $\Delta u = f$.*

1. *If $f \in L^q_k(\Omega)$ then $u \in L^q_{k+2}(K)$ or any compact set $K \Subset \Omega$, an if $u \in L^q(\Omega)$ then*

$$\|u\|_{L^q_{k+2}(K)} \leq C\left(\|\Delta u\|_{L^q_k(\Omega)} + \|u\|_{L^q(\Omega)}\right).$$

2. *(Schauder estimates). If $f \in C^{k,\alpha}(\Omega)$ then $u \in C^{k+2,\alpha}(K)$ for any compact subset $K \Subset \Omega$, and if $u \in C^{\alpha}(\Omega)$ then*

$$\|u\|_{C^{k+2,\alpha}(K)} \leq C\left(\|\Delta u\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^{\alpha}(\Omega)}\right).$$

Let us now generalize this theorem to Riemannian manifolds:

Theorem 2.7 (Global elliptic regularity)**.** *Let M be a compact Riemannian manifold, and suppose $u \in L^1_{loc}(M)$ is a weak solution to $\Delta u = f$.*

1. *If $f \in L^q_k(M)$, then $u \in L^q_{k+2}(M)$, and*

$$\|u\|_{q,k+2} \leq \ C\left(\|\Delta u\|_{q,k} + \|u\|_q\right).$$

2. *If $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$ and*

$$\|u\|_{C^{k+2,\alpha}} \leq C(\|\Delta u\|_{C^{k,\alpha}} + \|u\|_{C^{\alpha}}).$$

In the next three theorems we are interested to study the equation $(\Delta + h)u = 0$, where $h \in C^{\infty}(M)$ is a nonnegative function. This equation is sometimes called Schrödinger's equation, because of its similarity with the main equation of quantum mechanics. The following theorem is a classic for elliptic partial differential equations.

Theorem 2.8 (Strong maximum principle)**.** *Suppose h is a nonnegative, smooth function on a connected manifold M, and $u \in C^2(M)$ satisfies $(\Delta + h)u \geq 0$. If u attains its minimum $m \leq 0$, then u is constant on M.*

Before stating the next theorem we recall a common proposition from differential geometry and measure theory.

Proposition 2.9. *Any compact smooth Riemannian manifold (M, g) is a finite measure space when considering the volume form induced by the metric.*

Proof. For any point $p \in M$ we can find a chart on which every component of the metric is bounded and therefore the domain of the chart has finite measure. To see that such chart exists we can take any chart and then restrict it on an open set inside a compact neighborhood of the point which is contained in the original domain of the chart and then use Weierstrass theorem. Since the manifold is compact a finite number of such charts covers it and its total measure is bounded by the sum of the measures of the domains of its chart, which is finite and therefore the measure of the manifold is finite. □

The previous proposition allows us to use the comfortable theory of finite measure spaces. The following theorem gives us conditions to ignore removable singularities on weak solutions to Schrödinger's equation

Theorem 2.10 (Weak removable singularities theorem)**.** *Let U be an open set in M and $Q \in U$ and let h as before. Suppose u is a weak solution of $(\Delta + h)u = 0$ on $U \setminus \{Q\}$, with $h \in L^{n/2}(U)$ and $u \in L^q(U)$ for some $q > p/2 = n/(n-2)$. Then u satisfies $(\Delta + h)u = 0$ weakly on all of U.*

Proof. By definition of weak solution the thesis is equivalent to

$$\forall \varphi \in C_c^\infty(U) \qquad \int_U (u\Delta\varphi + hu\varphi)dV_g = 0,$$

where we used that Δ is self-adjoint. Now choose $\alpha \in C_c^\infty(U)$ with support in a ball $B_r(Q)$ such that $\forall y \in B_{r/2}(Q)$ $\alpha(y) = 1$ and define $\alpha_\varepsilon(x) = \alpha(x/\varepsilon)$ in normal coordinates around Q. Then $\mathrm{Supp}(\alpha_\varepsilon) \subseteq B_{\varepsilon r}(Q)$. Because $(1-\alpha_\varepsilon)\varphi$ is compactly supported in $U \setminus \{Q\}$ (indeed it vanishes in a whole neighborhood of Q) and in this set $(\Delta + h)u = 0$, it holds

$$\int_{U\setminus\{Q\}} (u\Delta(1-\alpha_\varepsilon)\varphi + hu(1-\alpha_\varepsilon)\varphi)dV_g = 0$$

which is equivalent to

$$\int_U (u\Delta\varphi + hu\varphi)dV_g = \int_U (u\Delta(\alpha_\varepsilon\varphi) + hu\varphi)dV_g.$$

Since the equality holds for every ε, by continuity we are done if we show that the right-hand side vanishes for $\varepsilon \to 0$.

Note that hu is integrable by Hölder's inequality. Indeed by Proposition 2.9 we know that our manifold is a finite measure space and therefore $u \in L^q(U) \subseteq L^{\frac{p}{2}}(U)$. Because

$$\left(\frac{n}{2}\right)^{-1} + \left(\frac{p}{2}\right)^{-1} = \frac{2}{n} + \frac{n-2}{n} = 1,$$

we get $hu \in L^1(U)$. Therefore, the second term above goes to zero as the support of α_ε shrinks. As for the first term, by the product rule of the Laplacian, which can be proved directly from its definition, we have

$$\Delta(\alpha_\varepsilon\varphi) = \varphi\Delta\alpha_\varepsilon - 2\langle\nabla\alpha, \nabla\varphi\rangle + \alpha_\varepsilon\Delta\varphi.$$

Moreover we have that $|\nabla\alpha_\varepsilon(x)| = |d(\alpha_\varepsilon)_x| = \frac{1}{\varepsilon}d\alpha_{x/\varepsilon} \leq \frac{C}{\varepsilon}$, where by $|d(\alpha_\varepsilon)_x|$ we mean its norm induced from the metric on the cotangent bundle. Indeed, since α is compactly supported, $d\alpha$ and $|d\alpha|$ are as well, and by Weierstrass theorem $|d\alpha| : x \mapsto |d\alpha_x|$ has a maximum C. Similarly $|\Delta\alpha_\varepsilon| \leq C/\varepsilon^2$. Therefore, if $\frac{1}{q} + \frac{1}{s} = 1$,

$$\begin{aligned}\left|\int_U u\Delta\left(\alpha_\varepsilon\varphi\right)dV_g\right| &\leq C\varepsilon^{-2}\int_{B_{\varepsilon r}} |u| dV_g \\ &\leq C\varepsilon^{-2}\left\|u\right\|_q \left(\int_{B_{\varepsilon r}} dV_g\right)^{\frac{1}{s}} \\ &\leq \; C'\varepsilon^{-2+n/s}\|u\|_q.\end{aligned}$$

Since $q > p/2$ implies $n/s > 2$, this goes to zero as $\varepsilon \to 0$.

Note that the example $u = r^{2-n}$ on Euclidean space shows that the hypothesis $q > p/2$ cannot be improved. □

3 The variational approach

3.1 The variational formulation

In this section we will turn the Yamabe problem to a variational problem. We will first derive some useful facts about conformal transformations. Recall from equation 1.3 that the scalar curvature transforms under conformal transformations according to the following law:

$$\tilde{S} = e^{-2f}\left(S + 2(n-1)\Delta_g f - (n-1)(n-2)|\nabla_g f|^2\right). \tag{3.1}$$

In this particular case, to simplify the formula, it is more convenient to make the substitution $e^{2f} = \varphi^{p-2}$ where $p = 2n/(n-2)$ and therefore $\tilde{g} = \varphi^{p-2}g$. With such an expedient the formula becomes

$$\tilde{S} = \varphi^{1-p}\left(4\frac{n-1}{n-2}\Delta_g\varphi + S\varphi\right),$$

which we can further simplify by defining the constants

$$p = \frac{2n}{n-2}, \quad a = 4\frac{n-1}{n-2} \tag{3.2}$$

and the differential operator

$$\Box_g = a\Delta_g + S \tag{3.3}$$

which is a differential operator in virtue of Proposition 1.3 and which is called *conformal Laplacian*. It is convenient to derive now how this operator transforms under conformal transformations, as it will come in handy in the future. The next proposition can be interpreted in some sense as a conformal covariance.

Proposition 3.1. *Let $g, \tilde{g}$ be two conformal metrics and let $\tilde{g} = \varphi^{p-2}g$, where p is defined above and φ is a smooth positive real function. Then for any smooth real function u*

$$\Box_{\tilde{g}}\left(\varphi^{-1}u\right) = \varphi^{1-p}\Box_g u.$$

Proof. By Corollary 1.10 we have that

$$\begin{aligned}\Delta_{\tilde{g}} &= \varphi^{2-p}\Delta_g + (n-2)\varphi^{2-p}g^{ij}\frac{\partial}{\partial x^j}\left(\frac{p-2}{2}\log\varphi\right)\frac{\partial}{\partial x^i} \\ &= \varphi^{2-p}\Delta_g + 2\varphi^{1-p}g^{ij}\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}.\end{aligned}$$

Combining this with Proposition 1.22 we have that

$$\begin{aligned}
\varphi^{p-1}\Box_{\tilde{g}}\left(\varphi^{-1}u\right) &= \varphi^{p-1}\left(a\Delta_{\tilde{g}}+\tilde{S}\right)u \\
&= a\varphi\Delta_g\left(\varphi^{-1}u\right)+2ag^{ij}\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}\left(\varphi^{-1}u\right)+\left(a\Delta_g\varphi+S\varphi\right)\left(\varphi^{-1}u\right) \\
&= a\Delta_g u-2a\left\langle \operatorname{grad}_g\left(\frac{u}{\varphi}\right),\operatorname{grad}_g\varphi\right\rangle - a\frac{u}{\varphi}\Delta_g u \\
&+2ag^{ij}\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}\left(\frac{u}{\varphi}\right)+a\frac{u}{\varphi}\Delta_g\varphi+Su \\
&= \Box_g u
\end{aligned}$$

which is equivalent of the above thesis. We made use of the following formula concerning the Laplacian, which can be proved from the product formula.

$$\Delta\left(\frac{f}{g}\right)=\frac{1}{g}\Delta f-\left\langle\frac{2}{g}\operatorname{grad}\left(\frac{f}{g}\right),\operatorname{grad}g\right\rangle-\frac{f}{g^2}\Delta g,\quad g\neq 0$$

Moreover we made use of the fact that

$$\left\langle\operatorname{grad}_g\left(\frac{u}{\varphi}\right),\operatorname{grad}_g\varphi\right\rangle=g^{ij}\frac{\partial\varphi}{\partial x^j}\frac{\partial}{\partial x^i}\left(\frac{u}{\varphi}\right),$$

which is true because the inner product of gradients equals the inner product of covariant derivatives and the coordinates of the covariant derivative are the partial derivatives for real functions. □

With the notation given by such new operator, $\tilde{g}=\varphi^{p-2}g$ has constant curvature λ if and only if

$$\lambda=\varphi^{1-p}\left(a\Delta_g\varphi+S\varphi\right),$$

that is to say

$$\Box_g\varphi=\lambda\varphi^{p-1}. \tag{3.4}$$

This last equation is called the *Yamabe equation*, and thanks to it we can reformulate the Yamabe problem as

Given a smooth compact Riemannian manifold (M,g) of dimension $n\geq 3$, does equation 3.4 have a solution?

Equation 3.4 is a sort of "nonlinear eigenvalue problem". Moreover it happens that if $q>p-1$ the methods based on linear theory are not usable for the equation $\Box_g\varphi=\lambda\varphi^q$. On the other hand if $q<p-1$ the problem can be brought to the linear case $q=1$ and can be easily solved. $q=p-1$ happens to be the critical value that separates the solvable case from the unsolvable, therefore it needs its own solution which will be more complex than the one of the linear case.

We are going to follow a variational approach.

Lemma 3.2. *Equation 3.4 is the Euler-Lagrange equation of the functional*

$$Q(\tilde{g})=\frac{\int_M\tilde{S}dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{2/p}} \tag{3.5}$$

over the set of metrics conformal to g.

Proof. We can rewrite $Q(\tilde{g}) = Q(\varphi^{-2}g) = Q_g(\varphi)$ where

$$Q_g(\varphi) = \frac{E(\varphi)}{\|\varphi\|_p^2} \tag{3.6}$$

$$E(\varphi) = \int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g \tag{3.7}$$

$$\|\varphi\|_p = \left(\int_M |\varphi|^p dV_g\right)^{1/p}. \tag{3.8}$$

We know that $\|\varphi\|_p$ exists since φ is smooth and M is compact. Integrating by parts the expression of $E(\varphi)$, we get to

$$\psi \in C^\infty(M) \quad \frac{d}{dt}Q_g\left(\varphi + t\psi\right)|_{t=0} = \frac{2}{\|\varphi\|_p^2}\int_M \left(a\Delta\varphi + S\varphi - \|\varphi\|_p^{-p}E(\varphi)\varphi^{p-1}\right)\psi dV_g,$$

which tells us, by the theory of calculus of variations that $\tilde{g}$ is a minimum if φ satisfies

$$a\Delta\varphi + S\varphi - \|\varphi\|_p^{-p}E(\varphi)\varphi^{p-1} = 0,$$

which is Yamabe equation for $\lambda = \|\varphi\|_p^{-p}E(\varphi)$. □

The functional Q_g defined in equation 3.6 in the proof above is more confortable than Q itself and we shall make use of it again in the future. It is usually referred to as the *Sobolev quotient* of φ on M. This is because it shifts the attention on the function φ instead of the metric, and this is convenient because the φ has the only contraint of being positive, whereas the metric $\tilde{g}$ has the more unusual contraint of being conformal to another metric. Moreover and more importantly φ contains all the information relevant to the problem and we are much more comfortable at working with functions than with symmetric 2-tensors. We can say something more about this functional

Proposition 3.3. *The functional* $Q : \{\tilde{g} \in \mathcal{T}^2(M) \mid \tilde{g}$ *metric on* M *conformal to* $g\} \to \mathbb{R}$ *defined above is bounded below.*

Proof. By Hölder's inequality, by choosing p' such that

$$\frac{2}{p} + \frac{1}{p'} = 1$$

we have that

$$\left|\int_M S\varphi^2\right| \leq \int_M \left|S\varphi^2\right| \leq \|S\|_{p'}\left\|\varphi^2\right\|_{\frac{p}{2}} \leq b\left\|\varphi\right\|_p^2$$

for some constant $b = \|S\|_{p'} \in \mathbb{R}^+$ (we know that $\|S\|_{p'}$ exists finite because S is a continuous function on a compact space) and hence

$$\varphi \mapsto \int_M S\varphi^2$$

is bounded below. Clearly

$$\varphi \mapsto \int_M a|\nabla\varphi|^2$$

is bounded below by 0 and hence Q_g is bounded below (see equations 3.6 and3.7). It follows that also Q is bounded below (being in fact the same thing). □

The last proposition allows us to introduce a central definition for the Yamabe problem.

Definition 3.1. We define the Yamabe invariant for any class $\{(M, g)\}$ of conformal metrics over the same compact manifold as

$$\begin{aligned}\lambda(M) &= \inf\{Q(\tilde{g}) \mid \tilde{g} \text{ conformal to } g\} \\ &= \inf\{Q_g(\varphi) \mid \varphi \text{ a smooth, positive function on } M\}.\end{aligned}$$

This constant determines whether the Yamabe problem can be solved or not. In chapter 3 we will prove that $\lambda(M) \leq \lambda(S^n)$, where S^n is the sphere with the same dimension as M. The combined work of Yamabe, Trudinger and Aubin showed that the Yamabe problem can be solved on any compact manifold M with $\lambda(M) < \lambda(S^n)$.

Note that the restriction to smooth positive test functions in Definition 3.1 above is unnecessary. Indeed by the Sobolev theorem the functional Q_g is continuous on the whole $L_1^2(M)$ space, and $C^\infty(M)$ is dense in such space. Moreover, $Q_g(|\varphi|) = Q_g(\varphi)$ for any smooth function φ, and a nonnegative function $(|\varphi|)$ can be approximated arbitrarily closely in L_1^2 norm by a positive function. Thus we may as well define

$$\lambda(M) = \inf\left\{Q_g(\varphi) \mid \varphi \in L_1^2(M)\right\}.$$

As anticipated the Yamabe problem has a solution for any compact manifold of dimension greater than three, but for some classes of manifolds it is solved much more easily than for others. In order to find a more significant characterization of manifolds where the Yamabe problem can be solved more easily, we need just a couple more definitions:

Definition 3.2. Let (M, g_M), (N, g_N) be two Riemannian manifold. A map $\phi : M \to N$ is said to be a *conformal map* if the pullback metric through ϕ is conformally equivalent to g_M, that is to say if $\phi^*(g_N) = \alpha g_M$ for some positive smooth function $\alpha : M \to \mathbb{R}^+$.

By unwrapping the definition we get that for all $v, w \in T_pM$ it holds that

$$g_N(d\phi_p(v), d\phi_p(w)) = \alpha g_M(v, w),$$

that is to say the map preserves angles, which is how we would naturally define it. Definition 3.2 is, however, more mathematically convenient.

Example. The stereographic projection of the sphere onto the plane with a point at infinity is a conformal map.

Interestingly, we do not need M and N to be Riemannian manifold to define the concept of conformal map. More precisely, we do not need lengths for conformal maps but only angles. How can we then define a structure that only defines angles? Taking inspiration from the theory of the Yamabe problem we define it as an equivalence class of metrics all conformal to each others. From the definition of conformal map it is clear that only the equivalence classes of metrics count in seeing whether or not the map is conformal. Therefore it makes sense to talk about conformal maps in the broader context of equivalence classes of metrics.

Definition 3.3. A Riemannian manifold M is said to be *locally conformally flat* if each point $x \in M$ has a neighborhood V which can be mapped to the flat space $\mathbb{R}^n$ with a conformal diffeomorphism. That is to say a there has to be a diffeomorphism $\phi : V \subseteq M \to \Omega \subseteq \mathbb{R}^n$ such that $\alpha \cdot g|_V = \phi^*(g_E)$ where g_E is the euclidean distance and α is a smooth positive function.

The following proposition is immediate

Proposition 3.4. *A Riemannian manifold (M, g) is locally conformally flat if and only if locally there is a conformal metric with null Riemann curvature tensor.*

Proof. If the manifold is locally conformally flat by definition there is locally an euclidean metric and hence curvature vanishes. The other implication follows from the theorem which states that a metric is Euclidean if and only if it is flat. □

Example. Every two-dimensional Riemannian manifold is locally conformally flat and for $n \geq 4$ any n-dimensional manifold is locally conformally flat if and only if the Weil tensor vanishes.

Taking for granted that the Yamabe problem can be solved on any compact manifold M such that $\lambda(M) < \lambda(S^n)$ and considering the definition of the constant $\lambda(M)$ as the inf of $Q_g(\phi)$, we just need to find a positive test function ϕ such that $Q_g(\phi) < \lambda(M)$. Aubin, precisely looking for such a "test function" was able to prove that for any compact manifold M of dimension greater than 6 that is not locally conformally flat we get $\lambda(M) < \lambda(S^n)$ and thus the Yamabe problem can be solved. The reason why this case can be solved more easily is that the test function can be built locally, compactly supported in a neighborhood of a point $p \in M$. This is the proof we will go through in this thesis.

In the remaining cases, that is to say for dimensions 3, 4, and 5 and for locally conformally flat manifolds, the local conformal geometry does not contain enough information to build the test function locally, so such a function must be built globally. The proof in such cases was found by Richard Schoen in 1984. Curiously Schoen's proof only works in the cases not covered by Aubin.

Semilinear partial differential equations with critical exponents such as the Yamabe equation 3.4 occur frequently in many contexts and this is the first time that such an equation was completely solved. For this reason, the Yamabe problem has a significant historical importance.

3.2 The sphere

In section 3.1 we saw that basically the whole Yamabe problems depends on the constant $\lambda(S^n)$ and how $\lambda(M)$ relates with it. Solving the Yamabe problem on the sphere is therefore essential for two reasons: first it is a key step in the solution of the general case, second it is a case whose manifold we know quite well, and which can inspire us one the general case. We already know a priori the solution to this particular case, as the sphere with its standard metric is a very basic example of a Riemannian manifold with positive constant scalar curvature. Therefore, the answer to the Yamabe problem for the sphere is positive, with no need of multiplying the metric tensor by a scalar function.

Anyway, it is still quite useful to solve the problem of the sphere with a variational approach, so to test our machinery on this case and derive the constant $\lambda(S^n)$ as a side result. Our next task will be therefore to prove that the standard metric on the sphere, up to a constant scale factor, is indeed the infimum of Yamabe's functional 3.5. The proof that this is the case was first found by Aubin in reference [Aub76] and independently by Talenti. Later on Uhlenbeck and Obata found a simpler proof which we now present with a simplification by Lee and Parker following a suggestion by Roger Penrose.

Proposition 3.5 (Obata)**.** *If g is a metric on S^n conformal to the standard metric $\overline{g}$ and with constant scalar curvature, then up to a constant scale factor, g is obtained from $\overline{g}$ by a conformal diffeomorphism of the sphere.*

Proof. To begin, we show that g is Einstein (see Definition 1.6). By hypothesis the two metrics are conformal, hence $\overline{g} = \frac{g}{\varphi^2}$ for some strictly positive function $\varphi \in C^\infty(S^n)$. Note that we are expressing the standard metric as a transformation of the new metric and not the converse, which is how we could have proceeded more intuitively. We can rewrite equation 1.3, that is the equation which tells us how the scalar curvature transforms, with the substitution $e^{2f} = \varphi^{-2}$, obtaining

$$\overline{R}_{jk} = R_{jk} + \varphi^{-1}\left((n-2)\varphi_{jk} - (n-1)\frac{\varphi_i\varphi^i}{\varphi}g_{jk} - \Delta_g\varphi g_{jk}\right).$$

Let $B_{jk} = R_{jk} - (S/n)g_{jk}$ represent the traceless Ricci tensor. Then, since the standard metric $\overline{g}$ is Einstein, it holds

$$0 = \overline{B}_{jk} = B_{jk} + (n-2)\varphi^{-1}\left(\varphi_{jk} + (1/n)\Delta\varphi g_{jk}\right).$$

Since the scalar curvature is constant, Lemma 1.23 implies that the divergence ${R^i}_{m,i}$ vanishes everywhere, and thus also does ${B^i}_{m,i}$. Since B is traceless, we can use integration by parts to get

$$\begin{aligned}\int_{S^n}\varphi|B|^2 dV_g &= \int_{S^n}\varphi B_{jk}B^{jk}dV_g \\ &= -(n-2)\int_{S^n}B^{jk}\left(\varphi_{jk} + \frac{1}{n}\Delta\varphi g_{jk}\right)dV_g \\ &= -(n-2)\int_{S^n}B^{jk}\varphi_{jk}dV_g \\ &= (n-2)\int_{S^n}{B^{jk}}_{,k}\varphi_j dV_g = 0.\end{aligned}$$

Thus B_{jk} must be identically zero and therefore g is an Einstein metric. Since g is conformal to the standard metric $\overline{g}$ on the sphere, which is locally conformally flat, we have $W = 0$ as well as $B = 0$. As noted in before, this implies that g has constant curvature, and so (S^n, g) is isometric to a standard sphere. The isometry is the desired conformal diffeomorphism. □

3.3 A bound for the Yamabe constant

In this section we will prove all connections between the Yamabe constant for a generic manifold and the Yamabe constant for the sphere that we mentioned in section 3.1. As a side effect we will find a more explicit form of the Yamabe constant for the sphere. Our goal is to minimize the functional Q_g defined in equation 3.6.

The first solution that might come to our mind, that is to say trying to find a sequence of functions for which the functional approaches its infimum, does not work. Indeed, suppose $\{u_i\}$ to be a sequence of smooth functions such that $Q_g(u_i) \to \lambda(M)$. By homogeneity we can assume $\|u_i\|_p = 1 \forall i$ (recall that $p = 2n/(n-2)$). We get

$$\begin{aligned}\|u_i\|_{2,1}^2 &= \int_M \left(|\nabla u_i|^2 + u_i^2\right) dV_g \\ &= \frac{1}{a} Q_g(u_i) + \int_M \left(1 - \frac{S}{a}\right) u_i^2 dV_g \\ &\leq \frac{1}{a} Q_g(u_i) + C \|u_i\|_p^2\end{aligned}$$

by Hölder's inequality. Therefore $\{u_i\}$ is bounded in $L_1^2(M)$. Bounded sets in a Hilbert space are weakly relatively compact (that is to say it is relatively compact with respect to the weak topology), and therefore the sequence u_i has a subsequence which converges weakly to a function $u \in L_1^2(M)$. However, as we saw, p is precisely the exponent for which the inclusion $L_1^2(M) \subseteq L^p(M)$ is not compact, we cannot know if $\|u_i\|_p = 1$ in the limit. Actually, the limit function u may happen to be identically zero.

We must adopt another approach. The following is an idea from Yamabe. Let us consider the perturbed functional

$$Q_g^s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2} \quad \text{for } 2 \leq s \leq p,$$

where $E(\varphi)$ was defined in equation 3.7. Analogously to what we did for Q_g we set

$$\lambda_s(M) = \inf \{Q^s(\varphi) : \varphi \in C^\infty(M)\}.$$

The Euler-Lagrange equation for this functional, with $\|\varphi\|_s = 1$, is

$$\Box_g \varphi = \lambda_s \varphi^{s-1} \tag{3.9}$$

where $\Box$ is the differential operator defined in 3.3. The subscript g is implied. To set things up, let us start with a basic regularity result for equation 3.9.

Theorem 3.6. *Suppose that $\varphi \in L_1^2(M)$ is a nonnegative weak solution to equation 3.9 with $2 \leq s \leq p$, and $\lambda_s \leq K$ for some constant K. If $\varphi \in L^r(M)$ for some $r > (s-2)\frac{n}{2}$ (in particular if $r = s < p$, or if $s = p < r$), then φ is either identically zero or strictly positive and C^∞, and $\|\varphi\|_{C^{2,\alpha}} \leq C$, where C depends only on M, g, K, and $\|\varphi\|_r$.*

Proof. If $\varphi \in L^r(M)$, by equation 3.9 we get that $a\Delta\varphi = \lambda_s \varphi^{s-1} - S\varphi \in L^q(M)$ with $q = \frac{r}{s-1}$. By elliptic regularity, Theorem 2.7, this implies that $\varphi \in L_2^q(M)$. The Sobolev embedding theorem now implies $\varphi \in L^{r'}(M)$ with

$$r' = \frac{nr}{ns - n - 2r}.$$

Because we supposed $r > (s-2)\frac{n}{2}$, we get $r' > r$. By iterating the argument we can show that $\varphi \in L_2^q(M)$ for all $q \in (1, +\infty)$. Now Theorem 2.4 implies that $\varphi \in C^\alpha(M)$ for some $\alpha \in \mathbb{R}^+$. Then $\varphi^{s-1} \in C^\alpha(M)$ as well, so by elliptic regularity we conclude that $\varphi \in C^{2,\alpha}(M)$. Each time we apply the Sobolev theorem above we get a bound on the corresponding norm, resulting in a final bound on $\|\varphi\|_{C^{2,\alpha}}$. From the equation $\square_g \varphi = \lambda_s \varphi^{s-1}$ it follows that $(\Delta + m)\varphi \geq 0$, for some constant $m \in \mathbb{R}^+$ and $m \geq \sup \frac{S - \lambda_s \varphi^{s-2}}{2}$. If $\varphi = 0$ somewhere, then $\varphi = 0$ everywhere by the strong maximum principle (Theorem 2.8). Thus φ is either strictly positive or identically 0. Finally, since φ is $C^{2,\alpha}$ and nowhere 0, so is φ^{s-1}, and so we can apply elliptic regularity iteratively to the equation $\square_g \varphi = \lambda_s \varphi^{s-1}$ to conclude that $\varphi \in C^\infty(M)$. □

The above result has been proven also in the borderline case $r = s = p$, but we do not need it in our approach. The following proposition is the reason why Yamabe's idea works.

Proposition 3.7. *For $2 \leq s < p$, there exists a smooth, positive solution φ_s to the subcritical equation 3.9, for which $Q^s_g(\varphi_s) = \lambda_s$ and $\|\varphi_s\|_s = 1$.*

Proof. Let $\{u_i\} \subset C^\infty(M)$ be a minimizing sequence for Q^s, with $\|u_i\|_s = 1$. Since $Q^s(|u_i|) = Q^s(u_i)$, after replacing u_i by $|u_i|$ we may assume $u_i \geqslant 0$. As we noted at the beginning of this section, $\{u_i\}$ is bounded in $L^2_1(M)$. Since the inclusion map $L^2_1 \subset L^s$ is compact, a subsequence of the $\{u_i\}$ converges weakly in L^2_1 and strongly in L^s to a function $\varphi_s \in L^2_1(M)$ with $\|\varphi_s\|_s = 1$. Since by Hölder's inequality the L^2 norm is dominated by the L^s norm, it follows that $\int S u_i^2 \to \int S \varphi_s^2$. Weak convergence in L^2_1 implies that

$$\begin{aligned}\int_M |\nabla \varphi_s|^2 \, dV_g &= \lim_{i\to\infty} \int_M \langle \nabla u_i, \nabla \varphi_s \rangle \, dV_g \\ &\leq \limsup_{i\to\infty} \left(\int_M |\nabla u_i|^2 \, dV_g \right)^{1/2} \left(\int_M |\nabla \varphi_s|^2 \, dV_g \right)^{1/2}\end{aligned}$$

and therefore $Q^s(\varphi_s) \leqslant \lim_{i\to\infty} Q^s(u_i) = \lambda_s$. But since λ_s is by definition the infimum of Q^s, we must have $Q^s(\varphi_s) = \lambda_s$, and so φ_s is extremal. Thus φ_s is a weak solution to the Euler-Lagrange equation 3.9. By the regularity theorem, φ_s is positive and C^∞. □

3.3.1 Back to the sphere

Before proceeding with the general theory, we are going to take advantage of the two previous theorems to conclude the case of the sphere.

Firstly we are going to prove a couple of lemmas which will make it easier to apply the Ascoli-Arzelà theorem in our context (where the metric space is a Riemannian compact manifold), which will turn out necessary more than one time. In the very first analysis courses we usually learn that given $a < b$ if $f : [a, b] \to \mathbb{R}$ is a continuous and differentiable function we have that

$$|f(b) - f(a)| \leq \sup_{x \in (a,b)} |f'(x)||b - a|.$$

This is because by the mean value theorem there exists $c \in (a, b)$ such that $(f(b) - f(a)) = f'(c)(b - a)$. Such an inequality is easily generalized to multiple valued functions defined on convex sets, where we replace the $f'(c)$ with $\text{grad}(f)(c)$ and c is meant to be a point lying on the segment joining the points we are considering.

Indeed this inequality can be generalized also to Riemannian manifolds.

Remark 3.1. Note first that given $f \in C^\infty(M)$, $\|\mathrm{grad}(f)(p)\| = \|\nabla f(p)\|$. Indeed recall from chapter 1 that $\nabla f = df$ and certainly $\|df_p\| = \|\mathrm{grad}(f)(p)\|$ because of the very way the inner product is extended to tensor bundles.

Theorem 3.8. *Let (M, g) be a complete Riemannian manifold, let $f \in C^\infty(M)$ and $x, y \in M$. Then there exists c on the support of the minimizing geodesic that connects x to y such that*

$$|f(x) - f(y)| \leq |\nabla f(c)| d(x, y)$$

where the distance is of course that induced by the metric tensor.

Proof. We know that the minimizing geodesic exists because the manifold is complete by hypothesis. Let $\ell = d(x, y)$ and $\gamma : [0, \ell] \to M$ be a unit speed minimizing geodesic with $\gamma(0) = x$ and $\gamma(\ell) = y$. Let $t_m \in [0, \ell]$ be the point where the function $t \mapsto |g\,(\mathrm{grad}\, f(\gamma(t)), \gamma'(t))|$ achieves its maximum. It exists because $[0, \ell]$ is compact. Define $c = \gamma(t_m)$. Then, by Proposition 1.7,

$$\begin{aligned}
|f(y) - f(x)| &= \left| \int_0^\ell g\left(\mathrm{grad}\, f(\gamma(t)), \gamma'(t)\right) dt \right| \\
&\leq \int_0^\ell \left| g\left(\mathrm{grad}\, f(\gamma(t)), \gamma'(t)\right) \right| dt \\
&\leq \int_0^\ell \left| g\left(\mathrm{grad}\, f\left(\gamma\left(t_m\right)\right), \gamma'\left(t_m\right)\right) \right| dt \\
&\leq |\,\mathrm{grad}\, f(c)| \int_0^\ell dt \\
&= |\,\mathrm{grad}\, f(c)| \ell = |\,\mathrm{grad}\, f(c)|\, d(x, y).
\end{aligned}$$

The last equality holds because γ is minimizing, which is a crucial hypothesis for the proof.

Finally, by the previous remark, $|\,\mathrm{grad}(f)(c)| = |\nabla f(c)|$, which concludes the proof. □

The following is just an adaptation of an analysis lemma to the context of Riemannian manifolds.

Lemma 3.9. *Let $\mathcal{F} = \{\varphi_s\}_{s \in S}$ be a family of real functions on a complete Riemannian manifold and let $k \in \mathbb{N}$. If $\mathcal{F}$ is bounded in $C^k_B(M)$ for $k \geq 1$ (see equation 2.1) then $\{\varphi_s\}_{s \in S}$ are equicontinuous.*

Proof. Fix $\epsilon \in \mathbb{R}^+$ and $x \in M$. We want to find $\delta \in \mathbb{R}^+$ independent of s such that $\forall s \in S\ \varphi_s(B_\delta(x)) \subseteq B_\varepsilon(\varphi_s(x))$.

Let $y \in M$. The fact that they are bounded in $C^k_B(M)$ tells us that there exists a constant $K \in \mathbb{R}^+$ such that $\forall s \in S\ \ \|\varphi_s\|_{C^k_B} \leq K$. In particular, since $k > 1$

$$\forall s \in S\ \sup_{x \in M} |\nabla \varphi_s| \leq K$$

and hence, by the previous lemma,

$$|\varphi_s(x) - \varphi_s(y)| \leq K|x - y|,$$

where K is valid for all $s \in S$. Take now $\delta = \frac{\varepsilon}{K}$. All $y \in B_\delta(x)$ satisfy

$$|\varphi_s(x) - \varphi_s(y)| \leq \varepsilon$$

and therefore $\varphi_s(y)$ belongs to $B_\varepsilon(\varphi_s(x))$. □

In particular the previous results hold for our compact manifold M since a compact metric space is also complete.

We now get back to our theory. The following theorem is perhaps the one with the hardest proof which we consider in this thesis. It is very important in the solution of the Yamabe problem as it will allow us to find out more about the constant $\lambda(S^n)$ and to make all connections between the Yamabe constant for the sphere and the Yamabe constant for a generic compact manifold, which as we anticipated is the key step in the solution to the general problem. The proof is due to Karen Uhlenbeck and makes use of a "renormalization" approach which will permit us, with a little more effort, to adapt the developed tools to prove the existence of extremals for the Sobolev quotient on the sphere. The argument which works for other manifolds does not work on the sphere because it uses the hypothesis that $\lambda(M) < \lambda(S^n)$ which clearly does not hold for the sphere (more deeply it is because on the sphere there is a compact group of conformal diffeomorphims).

Theorem 3.10. *There is a positive smooth function ψ on S^n satisfying $Q_{\overline{g}}(\psi) = \lambda(S^n)$.*

Proof. Let $s \in [0, p)$ and let φ_s be the solution to S^n to the subcritical problem 3.9 given by Proposition 3.7. We can suppose that the maximum of φ_s for each s is reached at the south pole, otherwise we just need to compose with a rotation. If the family $\{\varphi_s\}_s$ is uniformly bounded we can prove by Ascoli-Arzelà's theorem that there is a subsequence whose limit is a solution to the above equation just as we prove it for any other manifold in the rest of this thesis (see in particular Theorem 3.18). In such case the thesis is proved so suppose form now on that $\{\varphi_s\}_s$ is not uniformly bounded.

Now let $\kappa_\alpha = \sigma^{-1}\delta_\alpha\sigma : S^n \to S^r$ be the conformal diffeomorphism induced by the dilation δ_α on $\mathbb{R}^n$, as described when we talked about conformal diffeomorphisms on the sphere. If we set $g_\alpha = \kappa_\alpha^* \overline{g}$, where $\overline{g}$ is the standard metric on the sphere, we can write $g_\alpha = t_\alpha^{p-2}\overline{g}$, where the conformal factor t_α is the function

$$t_\alpha(\zeta, \xi) = \left(\frac{(1+\xi) + \alpha^2(1-\xi)}{2\alpha}\right)^{\frac{2-n}{2}},$$

where as always $(\zeta, \xi) = (\zeta^1, ..., \zeta^n, \xi)$ are coordinates on the sphere. Observe that at the south pole, $t_\alpha(0, -1) = \alpha^{\frac{2-n}{2}}$. For each $s \in [0, p)$, let $\psi_s = t_\alpha \kappa_\alpha^* \varphi_s$ and choose $\alpha = \alpha_s$ such that $\psi_s(0, -1) = 1$ (that is to say at the south pole). This implies that $\alpha_s = (\sup \varphi_s)^{\frac{2}{n-2}} \to +\infty$ as $s \to p$, and $\psi_s \leq \alpha^{(n-2)/2} t_\alpha$ on M. Let $\Box_{g_\alpha}$ denote the conformal Laplacian with respect to the metric g_α. By naturality (invariance with respect to the pull back) of the differential operator $\Box$, $\Box_{g_\alpha}(\kappa_\alpha^* \varphi_s) = \kappa_\alpha^*(\Box_{\overline{g}} \varphi_s)$. Then by the transformation law of $\Box$ found in Proposition 3.1

$$\Box_{\overline{g}} \psi_s = \Box(t_\alpha \kappa_\alpha^* \varphi_s) = t_\alpha^{p-1} \Box_{g_\alpha}(\kappa_\alpha^* \varphi_s) = \lambda_s t_\alpha^{p-1} (\kappa_\alpha^* \varphi_s)^{s-1} = \lambda_s t_\alpha^{p-s} \psi_s^{s-1}.$$

From this transformation we find that

$$\|\psi_s\|_{2,1} \leqslant C \int_{S^n} \psi_s \Box_{\overline{g}} \psi_s dV_{\overline{g}} = C \int_{S^n} \varphi_s \Box_{\overline{g}} \varphi_s dV_{\overline{g}} \leqslant C' \|\varphi_s\|_{2,1},$$

so $\{\psi_s\}$ is bounded in $L_1^2(S^n)$ with its norm, and so, by the Sobolev theorem, also in $L^p(S^n)$. Let $\psi \in L_1^2(S^n)$ denote the weak limit. Now let $P = (0,1)$ be the north pole. By Weierstrass theorem, on any compact subset of $S^n - \{P\}$ there exists a constant A such that $t_\alpha \leq A\alpha^{(2-n)/2}$, and therefore $\lambda_s t_\alpha^{p-s}\psi_s^{s-1}$, the right-hand side of the equation above, is bounded on $S^n - \{P\}$ by $\lambda_2 A^{p-1}$, independently of s. This implies that, on any compact set not containing the north pole, the right hand side is bounded in L^r for every r. $\{\psi_s\}$ is therefore bounded in $C^{2,\alpha}$ on compact sets disjoint from $\{P\}$, and therefore also in C^2 of the same sets.

That they are pointwise totally bounded follows easily from the fact that they are bounded in C^2. Moreover, by Lemma 3.9 they are also equicontinuous. Let now $K_1 \subseteq K_2 \subseteq ...$ be a sequence of compact sets whose union is $S^n \setminus \{P\}$. By Ascoli-Arzelà's theorem we can choose a subsequence $\{\psi_{1k}\}_{k\in\mathbb{N}}$ that converges in $C^2(K_1)$. Out of this subsequence, by Ascoli-Arzelà again, we may extract another subsequence $\{\psi_{2k}\}_{k\in\mathbb{N}}$ that converges in $C^2(K_2)$. By recurrence we have a sequence of subsequences $\{\psi_{ik}\}_{i,k\in\mathbb{N}}$ such that that $\{\psi_{ik}\}_{k\in\mathbb{N}}$ converges in $C^2(K_i)$. Now consider the sequence $\{\psi_{ii}\}_{i\in\mathbb{N}}$, that is to say the diagonal subsequence. This sequence has a limit function ψ as it has a limit on any compact set K_i and clearly the limit on a bigger compact agrees with that on a smaller K_i. Moreover, ψ is C^2 on $S^n \setminus \{P\}$ since ψ is C^2 on any compact and $\cup_i K_i = S^n \setminus \{P\}$.

Since $\lambda_s \to \Lambda$ and $t_\alpha^{p-s} \leq 1$ away from P for $s \to p$, we conclude that ψ satisfies $\Box_{\overline{g}}\psi = f\psi^{p-1}$ on $S^n - \{P\}$, for some C^2 function $f : M \to [0, \Lambda]$. By the weak removable singularities theorem, that is to say Theorem 2.10, $\Box_{\overline{g}}\psi = f\psi^{p-1}$ holds weakly on the whole S^n. For each s

$$\begin{aligned}\|\psi_s\|_p^p &= \int_{S^n} t_\alpha^p\,(\kappa_\alpha^*\varphi_s)^p\,dV_{\overline{g}}\\ &= \int_{S^n} (\kappa_\alpha^*\varphi_s)^p\,\kappa_\alpha^* dV_{\overline{g}} = \|\varphi_s\|_p^p \geq \mathrm{Vol}\,(S^n)^{1-p/s}\,\|\varphi_s\|_s^p\,.\end{aligned}$$

This implies that $\|\psi\|_p \geq 1$, and therefore $Q_{\overline{g}}(\psi) \leqslant \Lambda$. But since Λ is by definition the infimum of $Q_{\overline{g}}$, we must have $\Box_{\overline{g}}\psi = \Lambda\psi^{p-1}$ and $Q_{\overline{g}}(\psi) = \Lambda$.

It remains to show that ψ is positive and smooth. By the regularity theorem, it suffices to show that $\psi \in L^r(S^n)$ for some $r > p$. For $q > 1$ the operator $\Box_{\overline{g}} : L_2^q \to L^q$ has a bounded inverse by elliptic regularity. Consider the perturbation

$$\Box_\eta = \Box_{\overline{g}} - \eta\Lambda\psi^{p-2}$$

for $\eta \in C^\infty(S^n)$ supported in a small neighborhood of P. $\Box_\eta$ will also have a bounded inverse if the operator norm of the perturbation term $\eta\Lambda\psi^{p-2}$ is small enough. If we choose q such that $2n/(n+2) < q < n/2$ and set $r = nq/(n-2q)$, then for $u \in L_2^q$,

$$\left\|\eta\Lambda\psi^{p-2}u\right\|_q \leqslant \Lambda\left\|\eta^{1/p-2}\psi\right\|_p^{p-2}\|u\|_r \leqslant C\Lambda\left\|\eta^{1/(p-2)}\psi\right\|_p^{p-2}\|u\|_{q,2}$$

by the Hölder and Sobolev inequalities. Thus we can make the operator norm as small as we like by choosing η with small support and $0 \leqslant \eta \leqslant 1$. Now $\Box_\eta\psi = (1-\eta)\Lambda\psi^{p-1} \in L^q(S^n)$ since ψ is C^2 away from P. Therefore, since $\Box_\eta : L_2^q \to L^q$ is invertible, there exists $\xi \in L_2^q(S^n) \subset L_1^2(S^n)$ such that $\Box_\eta(\xi - \psi) = 0$. Using the Sobolev and Hölder inequalities as above, $\|u\|_{2,1}^2 \leqslant \int u\Box u \leqslant \int u\Box_\eta u + \varepsilon\|u\|_{2,1}^2$ when the support of η is small. Thus $\Box_\eta$ is injective on L_1^2, and so we must have $\psi = \xi \in L_2^q(S^n) \subset L^r(S^n)$. Since $r > p$, Theorem 3.6 implies that ψ is C^∞, and since $\psi = 1$ at the south pole, ψ is strictly positive. $\Box$

As a corollary we get the proof that the Yamabe problem on the sphere has a solution through our variational machinery.

Corollary 3.11. *The Yamabe functional on $(S^n, \overline{g})$ is minimized by constant multiples of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on S^n that have constant scalar curvature.*

Proof. It is a direct consequence of Proposition 3.5 and Theorem 3.10. □

Our next goal is to prove that for any compact manifold it holds $\lambda(M) \leq \lambda(S^n)$. In order to do this we will first rephrase the Yamabe problem for the sphere, which we already solved, on $\mathbb{R}^n$. This will lead to the sharp form of the Sobolev inequality - that is to say this will allow us to express the Sobolev constant in terms of $\lambda(M)$ and then we will make use of the extremals of the Sobolev inequality to prove $\lambda(M) \leq \lambda(S^n)$.

Of course, the main tool to translate to $\mathbb{R}^n$ the Yamabe problem on the sphere is the stereographic projection. Let $P = (0, \dots, 0, 1)$ be the north pole on $S^n \subset \mathbb{R}^{n+1}$. We can write down the stereographic projection explicitly as

$$\begin{aligned} \sigma : S^n - \{P\} &\to \mathbb{R}^n \\ \left(\zeta^1, \dots, \zeta^n, \xi\right) &\mapsto \left(x^1, \dots, x^n\right) \end{aligned} \qquad x^j = \frac{\zeta^j}{1-\xi}.$$

Let us prove that this is a conformal diffeomorphism. Indeed, if we denote with g_E the euclidean metric on $\mathbb{R}^n$ and with $\overline{g}$ the standard metric on S^n, we have the following relationship between the two metrics through the stereographic projection:

$$\sigma_* \overline{g} = 4\left(|x|^2 + 1\right)^{-2} g_E$$

where $\sigma_* = (\sigma^{-1})^*$ denotes the push-forward of a tensor field. We rewrite this as

$$\sigma_* \overline{g} = 4 u_1^{p-2} g_E \qquad u_1(x) = \left(\frac{1}{|x|^2 + 1}\right)^{\frac{n-2}{2}}.$$

We can exploit the stereographic projection and the knowledge of conformal diffemorphisms in euclidean space to write down all conformal diffeomorphisms on the sphere. Indeed, up to composing the map with a rotation or, if you prefer, the antipodal map, the diffeomorphism must have a fixed point, which we might think as our north pole. This shows us that we can think the sphere's conformal diffeomorphism acting on $\mathbb{R}^n$. The group of such transformations is generated by rotation and by maps of the form $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\alpha\sigma$ where τ_v is a translation by a vector $v \in \mathbb{R}^n$

$$\tau_v(x) = x - v$$

and δ_α is a dilation by a factor $\alpha^{-1} \in \mathbb{R}^+$

$$\delta_\alpha(x) = \alpha^{-1} x.$$

Under dilations, the spherical metric transforms as

$$\delta_\alpha^* \sigma_* \overline{g} = 4 u_\alpha^{p-2} g_E, \qquad \text{where } u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{\frac{n-2}{2}}. \tag{3.10}$$

In the following theorem we are going to make use of a bump function (which is a classic in differential geometry), that is to say, given $\varepsilon \in \mathbb{R}^+$, a radial (that depends only on $r = |x|$) function $\eta_\varepsilon : \mathbb{R}^n \to [0,1]$ such that its support is contained in $B_{2\varepsilon}(0)$, and such that $\eta_\varepsilon \equiv 1$ in $B_\varepsilon(0)$. In this particular case we are going to use the letter M (not to be confused with the manifold) instead of ε because we will take the limit to ∞ instead of 0, but the zero limit is more common. We can now prove

Theorem 3.12. *The n-dimensional Sobolev constant σ_n is equal to $\frac{a}{\Lambda}$, where*

$$\Lambda = \lambda(S^n) = Q(\overline{g}) = n(n-1)\operatorname{vol}(S^n)^{2/n}$$

and, as defined at the beginning of this section, $a = 4\frac{n-1}{n-2}$. Thus the Sobolev inequality 2.2 on $\mathbb{R}^n$ becomes

$$\|\varphi\|_p^2 \leqslant \frac{a}{\Lambda}\int_{\mathbb{R}^n} |\nabla\varphi|^2 dx.$$

Equality is attained only by constant multiples and translates of the functions u_α defined above.

Proof. For any $\varphi \in C^\infty(S^n)$ define $\overline{\varphi} = u_1\sigma_*\varphi$, that is to say the weighted push-forward function on $\mathbb{R}^n$, where u_1 is defined above. Thus we have

$$\sigma_*\left(\varphi^{p-2}\overline{g}\right) = 4\overline{\varphi}^{p-2}g_E.$$

By conformal invariance of Q, we have that

$$\lambda(S^n) = \inf_{\varphi\in C^\infty(S^n)} \frac{\int_{\mathbb{R}^n} a|\nabla\overline{\varphi}|^2 dx}{\left(\int_{\mathbb{R}^n} |\overline{\varphi}|^p dx\right)^{2/p}}.$$

Now we are going to use a cutoff function argument that we are going to use many other times in this thesis. Let $\eta_M : \mathbb{R}^n \to [0,1]$ be a smooth bump function such that $\eta_M \equiv 1$ in $B_M(0)$ and $\eta_M \equiv 1$ on $\mathbb{R}^n \backslash B_{2M}(0)$. Up to smooth things out by introducing a couple of ε-wide rings we can suppose $|\nabla\eta_M(x)| = \frac{1}{M}$ for all $x \in A_M = B_{2M}(0) \setminus B_M(0)$. Let $\tilde{\varphi} = \eta_M\overline{\varphi} = u_1\sigma_*\varphi\eta_M$. We have, by denoting $\hat{\varphi} = \sigma_*\varphi$ and $\eta = \eta_M$,

$$\begin{aligned}\int_{\mathbb{R}^n} a|\nabla\tilde{\varphi}|^2 dx &= \int_{B_{2M}} \left(a\eta^2\left|\nabla\left(\hat{\varphi}u_1\right)\right|^2 + 2a\eta\hat{\varphi}u_1\left\langle\nabla\eta, \nabla\left(\hat{\varphi}u_1\right)\right\rangle + a\hat{\varphi}^2u_1^2|\nabla\eta|^2\right)dx \\ &\leq \int_{\mathbb{R}^n} a\left|\nabla\overline{\varphi}\right|^2 dx + C\int_{A_M}\left(\frac{2}{M}\overline{\varphi}\left|\partial_r\overline{\varphi}\right| + \frac{1}{M^2}\overline{\varphi}^2\right)dx.\end{aligned}$$

Because $\hat{\varphi}$ has a finite limit at infinity (it is the push-forward of a function on the sphere) we have that

$$\int_{\mathbb{R}^n} a|\nabla\tilde{\varphi}|^2 dx \leq \int_{\mathbb{R}^n} a\left|\nabla\overline{\varphi}\right|^2 dx + \tilde{C}\int_{A_M}\left(\frac{2}{M}u_1\left|\partial_r u_1\right| + \frac{1}{M^2}u_1^2\right)dx.$$

Now we need to estimate the term on the right. Suppose $M > 1$ We have that

$$u_1(r) = \left(\frac{1}{r^2+1}\right)^{\frac{n-2}{2}} \leq \left(\frac{1}{M^2+1}\right)^{\frac{n-2}{2}} \leq \left(\frac{2}{M}\right)^{n-2}.$$

Moreover, a simple derivation shows that

$$|\partial_r u_1| = \left|(2-n)r\left(\frac{1}{r^2+1}\right)^{\frac{n}{2}}\right| \leq 2(n-2)M\left(\frac{2}{M}\right)^{n-2} = 4(n-2)\left(\frac{2}{M}\right)^{n-1}.$$

Thus

$$\tilde{C}\int_{A_M}\left(\frac{2}{M}u_1\left|\partial_r u_1\right|+\frac{1}{M^2}u_1^2\right)dx\leq\overline{C}\left(\frac{1}{M}\right)^{2n}\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\left((2M)^n-M^n\right)\leq\hat{C}\frac{1}{M^n}.$$

This shows that we may restrict ourselves to functions with compact support on $\mathbb{R}^n$ as for every non compactly supported function, by what we saw, we may find a sequence of function whose infimum is the same as that of the non compactly supported function. Therefore

$$\lambda\left(S^n\right)=\inf_{\varphi\in C_C^\infty(\mathbb{R}^n)}\frac{a\|\nabla\varphi\|_2^2}{\|\varphi\|_p^2}.$$

By recalling the Sobolev inequality 2.2

$$\|\phi\|_p^2\leq\sigma_n\int_{\mathbb{R}^n}|\nabla\phi|^2\,dx,$$

by the fact that $C_C^\infty(\mathbb{R}^n)$ is dense in $L_1^2(\mathbb{R}^n)$ and by definition of infimum we get the thesis. The fact that equality is attained only by constant multiples and translates of u_α and the explicit expression of Λ follow from Corollary 3.11. □

Before proving the goal result of this section we need just one more technical calculus lemma, which we state in its greater generality as we will need it again later.

Lemma 3.13. *Let $k\in\mathbb{Z}$ and suppose $k>-n$. Moreover, let*

$$I(\alpha)=\int_0^\varepsilon r^k u_\alpha^2 r^{n-1}dr.$$

It holds that for $\alpha\to 0$ $|I(\alpha)|<cf(\alpha)$ where $c\in\mathbb{R}^+$ and

$$f(\alpha)=\begin{cases}\alpha^{k+2} & \text{if } n>k+4\\ \alpha^{k+2}\log\left(\frac{1}{\alpha}\right) & \text{if } n=k+4\\ \alpha^{n-2} & \text{if } n<k+4\end{cases}$$

Proof. The substitution $\sigma=r/\alpha$ in the integral above gives

$$I(\alpha)=\alpha^{k+2}\int_0^{\varepsilon/\alpha}\sigma^{k+n-1}\left(\sigma^2+1\right)^{2-n}d\sigma.$$

Observe that $\sigma^2\leq\sigma^2+1\leq 2\sigma^2$ for $\sigma\geq 1$, so $I(\alpha)$ is bounded above and below by positive multiples of

$$\alpha^{k+2}\left(C+\int_1^{\varepsilon/\alpha}\sigma^{k+3-n}d\sigma\right).$$

Expanding asymptotically the expression in parenthesis we have the thesis. □

Even if $\mathbb{R}^n$ is not a compact manifold, it still holds that $\lambda(\mathbb{R}^n)\leq\lambda(S^n)$. We will start proving this, which is easier than on a generic compact Riemannian manifold, and then we will extend the result as we want it.

Lemma 3.14. *Let $\varepsilon \in \mathbb{R}^+$ and let $\varphi = u_\alpha \cdot \eta_\varepsilon$ where u_α are the functions defined in equation 3.10 and η_ε is the bump function defined right before Theorem 3.12. Then φ is a smooth radial function compactly supported in $B_{2\varepsilon}(0)$ and it holds that, if $n \geq 3$,*

$$\forall \alpha \in \mathbb{R}^+ \; Q_g(\varphi) \leq \Lambda + C\alpha^{n-2} \tag{3.11}$$

for an appropriate C, where g is the standard metric on $\mathbb{R}^n$. Therefore, as a result $\lambda(\mathbb{R}^n) \leq \lambda(S^n)$.

Proof. This proof is going to be very similar to that of Theorem 3.12. That φ is a smooth radial function compactly supported in $B_{2\varepsilon}(0)$ is clear from the fact that it is a product of two smooth radial functions one of which is compactly supported in $B_{2\varepsilon}(0)$. The functions u_α satisfy $a \left\|\nabla u_\alpha\right\|_2^2 = \Lambda \left\|u_\alpha\right\|_p^2$ on $\mathbb{R}^n$. Since φ is a function of $r = |x|$ alone, we have that

$$\begin{aligned} E(\varphi) = \int_{\mathbb{R}^n} a|\nabla\varphi|^2 dx &= \int_{B_{2\varepsilon}} \left(a\eta^2 \left|\nabla u_\alpha\right|^2 + 2a\eta u_\alpha \left\langle \nabla\eta, \nabla u_\alpha \right\rangle + a u_\alpha^2 |\nabla\eta|^2\right) dx \\ &\leq \int_{\mathbb{R}^n} a \left|\partial_r u_\alpha\right|^2 dx + C \int_{A_\varepsilon} \left(u_\alpha \left|\partial_r u_\alpha\right| + u_\alpha^2\right) dx \end{aligned} \tag{3.12}$$

where A_ε denotes the anulus $B_{2\varepsilon} - B_\varepsilon$. We want to estimate both addends in the right side of the last inequality. Because $n \geq 3$

$$u_\alpha = \left(\frac{\alpha}{|x|^2 + \alpha^2}\right)^{\frac{n-2}{2}} \leq \left(\frac{\alpha}{|x|^2}\right)^{\frac{n-2}{2}} = \alpha^{\frac{n-2}{2}} r^{2-n}.$$

Moreover, a simple derivation shows that

$$\partial_r u_\alpha = (2-n) r \alpha^{-1} \left(\frac{r^2 + \alpha^2}{\alpha}\right)^{-\frac{n}{2}}$$

which, combined with the previous inequality provides the estimate

$$\left|\partial_r u_\alpha\right| \leqslant (n-2)\alpha^{(n-2)/2} r^{1-n}.$$

Thus the second term in the inequality 3.12 is $O(\alpha^{n-2})$ as $\alpha \to 0$. For the first we have the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} a \left|\partial_r u_\alpha\right|^2 dx &= \Lambda \left(\int_{B_e} u_\alpha^p dx + \int_{\mathbb{R}^n - B_\varepsilon} u_\alpha^p dx\right)^{2/p} \\ &\leqslant \Lambda \left(\int_{B_{2\varepsilon}} \varphi^p dx + \int_{\mathbb{R}^n - B_e} \alpha^n r^{-2n} dx\right)^{2/p} \\ &= \Lambda \left(\int_{B_{2\varepsilon}} \varphi^p dx\right)^{2/p} + O\left(\alpha^n\right) \\ &= \Lambda \left\|\varphi\right\|_p^2 + O\left(\alpha^n\right) \end{aligned}$$

And so

$$Q_g(\varphi) \leq \Lambda + \tilde{C}\alpha^{n-2} + O(\alpha^n)$$

for an appropriate $\tilde{C}$. For α small enough we can forget about the term $O(\alpha^n)$ by slightly modifying $\tilde{C}$, thus proving the thesis. □

Finally we can prove our goal result, due to Aubin, in the general case of compact manifolds

Theorem 3.15. *For any compact manifold M of dimension $n \geq 3$ it holds that $\lambda(M) \leq \lambda(S^n)$.*

Proof. By the previous lemma we can choose $\varphi = \eta \cdot u_\alpha$ as defined before so that it has support in $B_{2\varepsilon}(0)$ for any $\varepsilon \in \mathbb{R}^+$. We choose ε small enough so that $B_{2\varepsilon}(0)$ is contained in a domain of a normal coordinate parametrization x^i around $P \in M$. Such a φ is canonically extended to a real smooth function on the manifold, by defining it as it is on the normal coordinates chart and by extending it to 0 on the rest of the manifold. Because φ is a radial function, and $g^{ii} = 1$ in normal coordinates, we have $|\nabla\varphi|^2 = |\partial_r\varphi|^2$ as in the previous lemma. The only correction we have to make to the estimate 3.12 is that we have to insert the term in the scalar curvature in the definition of $E(\varphi)$ and we have to take into account the difference between dV_g and dx. But since we are in normal coordinates, $dV_g = (1 + O(r))dx$ and the previous calculation becomes

$$
\begin{aligned}
E(\varphi) &= \int_{B_{2e}} \left(a|\nabla\varphi|^2 + S\varphi^2\right) dV_g \\
&\leqslant (1 + C\varepsilon)\left(\Lambda\|\varphi\|_p^2 + C\alpha^{n-2} + C\int_0^{2\varepsilon}\int_{S_r} u_\alpha^2 r^{n-1} d\omega dr\right).
\end{aligned}
$$

By Lemma 3.13 we have that the last term is bounded by a constant multiple of α. Thus by choosing first ε and then α small enough, we can prove

$$
Q_g(\varphi) \leqslant (1 + C\varepsilon)(\Lambda + C\alpha).
$$

In the limit of $\varepsilon \to 0$ and $\alpha \to 0$ we have $\lambda(M) \leq \Lambda$. □

3.3.2 Sufficient condition for the existence of a solution

Our next task is to prove that the Yamabe problem on a compact manifold M has a solution provided that $\lambda(M) < \lambda(S^n)$.

To do so we will make use of the functions φ_s which we found in Proposition 3.7. Indeed we saw that they are a solution for the subcritical equation 3.9 and therefore our hope is that they are a solution to the Yamabe equation in the limit $p \to s$. Indeed, we are about to find out that this is the case and that the convergence is uniform. As a preamble, note that by multiplying g by a constant (which is a conformal transformation and therefore does not change the conformal class), we can reduce ourselves to consider $\int_M dV_g = 1$, which slightly simplifies calculations. In the next lemma we study the behavior of λ_s in the limit $s \to p$.

Lemma 3.16. *If $\int_M dV_g = 1$, then the function*

$$
\begin{aligned}
[0, p] &\to \mathbb{R}^+ \\
s &\mapsto |\lambda_s|
\end{aligned}
$$

is nonincreasing and if $\lambda(M) \geq 0$ it is continuous from the left.

Proof. To see that $s \mapsto |\lambda_s|$ is nondecreasing observe that for any s and s' and a nonzero $u \in C^\infty(M)$ it holds that

$$Q^{s'}(u) = \frac{\|u\|_s^2}{\|u\|_{s'}^2} Q^s(u)$$

By Hölder's inequality, if $s \leq s'$, then $\|u\|_s \leq \|u\|_{s'}$ and therefore $|\lambda_{s'}| \leq |\lambda_s|$. This proves that the function defined above is nondecreasing.

By standard analysis results, if $\lambda_s < 0$ for some s, there exists $u \in C^\infty(M)$ such that $Q^s(u) < 0$ and by what we just saw, the ratio between $Q^s(u)$ and $Q^{s'}(u)$ is a positive constant, therefore $Q^{s'}(u) < 0$ for any s'. Thus $\lambda_s < 0$ for all s.

This also proves that if instead $\lambda(M) \geq 0$ it must be that $\lambda_s \geq 0$ for any $s \in [2, p]$. So take $s \in [2, p]$ and take $\varepsilon \in \mathbb{R}^+$. Then by definition of inf there exists a u such that $Q^s(u) < \lambda_s + \varepsilon$. Since $\|u\|_s$ is a continuous function of s, there exists a δ sufficiently small such that for all $s' \in [s - \delta, s]$ it holds that $\lambda_{s'} \leqslant Q^{s'}(u) < \lambda_s + 2\varepsilon$. Combining this with the fact that λ_s is nonincreasing, we have that λ_s is continuous from the left. □

We need one last more small step before the goal result

Lemma 3.17. *Suppose $\lambda(M) < \lambda(S^n)$, and let $\{\varphi_s\}$ be the set of solution to $\Box\varphi = \lambda_s \varphi^{s-1}$ such that $Q_g^s(\varphi_s) = \lambda_s$ and $\|\varphi_s\|_s = 1$, as defined in Proposition 3.7. Then we have constants $s_0 < p$, $r > p$ and $C > 0$ such that $\|\varphi_s\|_r \leq C$ for all $s \geq s_0$, that is to say it is bounded independently from s.*

Proof. By Proposition 3.7 we know that the set $\{\varphi_s\}$ is nonempty. Fix $\delta \in \mathbb{R}^+$. By multiplying our equation $\Box_g \varphi = \lambda_s \varphi^{s-1}$ by $\varphi_s^{1+2\delta}$ and integrating, we get

$$\begin{aligned}
\int_M \varphi_s^{1+2\delta} \, \Box_g \varphi_s dV_g &= \int_M \left(a\varphi_s^{1+2\delta} \Delta_g \varphi_s + S\varphi_s^{2+2\delta} \right) dV_g \\
&= \int_M \left(a \left\langle d\varphi_s, (1+2\delta)\varphi_s^{2\delta} d\varphi_s \right\rangle + S\varphi_s^{2+2\delta} \right) dV_g \\
&= \lambda_s \int_M \varphi_s^{s+2\delta} dV_g
\end{aligned}$$

where, in the second equality, we made use of Proposition 1.8.

To simplify notation, define $w = \varphi_s^{1+\delta}$. The last equality, by subtracting the term involving S on both sides, becomes

$$\frac{1+2\delta}{(1+\delta)^2} \int_M a|dw|^2 dV_g = \int_M \left(\lambda_s w^2 \varphi_s^{s-2} - Sw^2 \right) dV_g$$

Now, by the sharp Sobolev inequality for manifolds (that is to say that of Theorem 2.5 with the optimum constant found in Theorem 3.12), we have that for any $\varepsilon \in \mathbb{R}^+$ there exists C_ε such that

$$\begin{aligned}
\|w\|_p^2 &\leq (1+\varepsilon)\frac{a}{\Lambda} \int_M |dw|^2 dV_g + C_\varepsilon \int_M w^2 dV_g \\
&\leq (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \int_M \frac{\lambda_s}{\Lambda} w^2 \varphi_s^{s-2} dV_g + \left(C_\varepsilon - (1+\varepsilon)\frac{a}{\Lambda}\frac{(1+\delta)^2}{(1+2\delta)} S \right) \|w\|_2^2 \\
&\leq (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \frac{\lambda_s}{\Lambda} \|w\|_p^2 \, \|\varphi_s\|_{(s-2)n/2}^{s-2} + C'_\varepsilon \|w\|_2^2
\end{aligned}$$

where the definition of C_ε is clear from the context. The last inequality follows form Hölder's inequality.

Since $s < p$

$$\frac{(s-2)n}{2} < s$$

Thus, we can use Hölder's inequality a second time to deduce that $\|\varphi_s\|_{(s-2)n/2} \leq \|\varphi_s\|_s = 1$. Now we are going to consider two cases separately, that is to say $\lambda(M) \geq 0$ and $\lambda(M) < 0$.

If $0 \leq \lambda(M) \leq \lambda(S^n)$, then for some $s_0 < p$,

$$\frac{\lambda_s}{\Lambda} \leq \frac{\lambda_{s_0}}{\Lambda} \leq 1$$

for $s > s_0$. Thus, since the function

$$G(\varepsilon, \delta) = (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)}$$

is such that $G(0,0) = 1$, by continuity we can choose ε and δ in $\mathbb{R}^+$ small enough so that the coefficient of the first term above

$$G' = (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)}\frac{\lambda_s}{\Lambda}\|\varphi_s\|^{s-2}_{(s-2)n/2}$$

is less than 1. Therefore

$$\|w\|_p^2(1-G') \leq C'_\varepsilon \|w\|_2^2$$

and therefore we have a positive constant $C = \frac{C'_\varepsilon}{(1-G')}$ such that

$$\|w\|_p^2 \leqslant C' \|w\|_2^2.$$

The same result clearly holds if $\lambda(M) < 0$ (and hence $\lambda_s < 0$).

Now, in both cases, by Hölder's inequality

$$\|w\|_2 = \|\varphi_s\|^{1+\delta}_{2(1+\delta)} \leq \|\varphi_s\|^{1+\delta}_s = 1,$$

therefore $\|w\|_p = \|\varphi_s\|^{1+\delta}_{p(1+\delta)}$ is such that

$$\|\varphi_s\|^{1+\delta}_{p(1+\delta)} \leq C'$$

and hence the thesis follows by choosing $r = p(1+\delta) > p$ and $C = C'^{\frac{1}{1+\delta}} > 0$. □

Finally we are able to prove that the Yamabe problem has solution if $\lambda(M) < \lambda(S^n)$.

Theorem 3.18. *Let $\{\varphi_s\}$ be as in the previous theorem and suppose $\lambda(M) < \lambda(S^n)$. As $s \to p$ there exists a subsequence of such functions that converges uniformly to a positive function $\varphi \in C^\infty(M)$ such that*

$$Q_g(\varphi) = \lambda(M) \qquad \Box_g \varphi = \lambda(M)\varphi^{p-1}$$

Thus the metric $\tilde{g} = \varphi^{p-2} g$ has constant scalar curvature.

Proof. Since the functions $\{\varphi_s\}$ are uniformly bounded in $L^r(M)$, by the regularity Theorem 2.7 we know that they are uniformly bounded in $C^{2,\alpha}(M)$ as well. Since real numbers are a complete metric space and our manifold M is supposed to be compact in the whole thesis, we can apply Ascoli-Arzelà theorem if we prove the functions to be equicontinuous and pointwise totally bounded. $\{\varphi_s\}$ are uniformly bounded and therefore pointwise totally bounded. Moreover, since $C^{2,\alpha}(M) \subseteq C^2(M)$ they are equicontinuous by Lemma 3.9. Hence, Ascoli-Arzelà tells us that there is a subsequence that converges uniformly to a function φ. Actually we have more: similarly to what happens in $\mathbb{R}$, with the same proof, if the functions are in $C^k(M)$, the subsequence converges in norm C^k, so in our case it converges in norm C^2 to $\varphi \in C^2(M)$. In virtue of this fact, because the differential operator $\Box$ and the functional Q_g make use of at most second derivatives, they commute with the limit and therefore φ satisfies

$$\Box\varphi = \lambda\varphi^{p-1}, \quad Q_g(\varphi) = \lambda$$

where $\lambda = \lim_{s\to p} \lambda_s$. If $\lambda \geq 0$, by Lemma 3.16 we have that $\lambda = \lambda(M)$ by continuity on the left. If, on the other hand, $\lambda(M) < 0$, the fact that λ_s is increasing implies $\lambda \leq \lambda(M)$, but since $\lambda(M)$ is the infimum we have that $\lambda = \lambda(M)$ in this case as well.

Finally, Theorem 3.6 proves that φ is C^∞ and it is strictly positive since $\|\varphi\|_p \geq \lim_{s\to p} \|\varphi_s\|_s = 1$ (and therefore it cannot be identically zero).

□

4 Conformal normal coordinates

In this chapter we are going to introduce a tool which was not present in the first proofs of the solution to the Yamabe problem, but was firstly introduced by Parker and Lee in reference [LP87] which is our main reference. We are talking about conformal normal coordinates, which are a variation of the common normal coordinates of differential geometry, and which make the solution of the problem easier to follow and more elegant.

Indeed, in mathematics it often happens that a newly found solution to a problem or a proof of a theorem is long, technical and somehow incomprehensible. In such cases the job of a mathematician is not finished but in reality it just begun. At this point it is often not a matter of having smart ideas to simplify the existing proof, but it is more a matter of being creative and develop new theory around the problem so to find radically different proofs and transform a long a tortuous solution into a sequence of short and obvious lemmas and corollaries so that the whole thing looks way more natural. In other word, at this point, it becomes a matter of working with definitions rather than with proofs.

This is exactly what we are trying to do with conformal normal coordinates. Thus, this chapter will not be devoted directly to the solution of our problem, but rather to the development of the theory around such new coordinates so to shorten significantly the proof of the Yamabe problem and work with something which has a rather more general interest.

The main idea of such new coordinates goes back to what we were talking about in chapter 1, that is to say we were saying that to define the concept of angles on a smooth manifold we do not need a Riemannian metric, but rather a conformal class of Riemannian metrics. Namely, we quotient the set of metrics on a smooth manifold modulo the equivalence relation by which two metrics g and g' are equivalent if there exists a smooth positive real function u such that $g' = ug$.

In differential geometry normal coordinates exploit the huge freedom in choosing a chart for a neighborhood of a point to find a chart which makes the local geometry of a manifold look similar to that of $\mathbb{R}^n$. We are going to proceed similarly, but in our case we do not care about the metric being preserved (indeed it makes no sense to talk about a metric if we only have a conformal class of metrics) but only the angles. This provides us with additional freedom: we can choose the geodesic normal coordinates of a particular metric in our conformal class so to simplify the local geometry even more than normal coordinates in a fixed Riemannian structure.

Normalized conformal structures were first constructed by Robin Graham, who proved, in connection with his work with Charles Fefferman on conformal invariant theory, that for any $p \in M$ it is possible to find a conformal metric for which the symmetrized covariant derivatives of the Ricci tensor of g vanish at p. Graham's normalization is designed to simplify the algebra of conformal invariant theory. For our purposes, we want a normalization that simplifies the local analysis.

We can get inspiration (and maybe this is how the authors of our reference had the idea) from the expression of the Laplacian in coordinates in equation 1.1. It is clear that the formula would be greatly simplified if we chose a conformal metric for which $\det g = 1$ in g-normal coordinates. Intrinsically, this means that the Jacobian of the exponential map of g is 1. As we are about to see, it turns out that this is the normalization that implements the desired simplification.

Our next goal is to prove that in each conformal class of metrics there is indeed a metric that has the desired property, which is by no means trivial.

We begin with a technical lemma, necessary for a key theorem in the proof our goal theorem. Consider $p \in M$ and let x^i be g-normal coordinates on a neighorhood of p. We will denote by $\mathbb{R}[x^1, ..., x^n]_m = \mathbb{R}[x]_m$ the set of real homogeneous polynomials of degree m in g-normal coordinates. Given $x = (x^1, ..., x^n)$ we will also denote with r its norm $r = |x|$ and finally we denote by Δ_0 the euclidean Laplacian in x-coordinates, that is to say

$$\Delta_0 = \sum_i \frac{\partial^2}{(\partial x^i)^2}.$$

Lemma 4.1. *The eigenvalues of the differential operator $r^2\Delta_0$ on the vector space $\mathbb{R}[x]_m$ are*

$$\{\lambda_j = -2j(n-2+2m-2j) : j = 0, \dots, \lfloor m/2 \rfloor\}$$

The eigenvectors corresponding to λ_j are the homogeneous polynomials of the form $r^{2j}u$ where $u \in \mathbb{R}[x]_{m-2j}$ is an homogeneous polynomial.

Proof. We are going to proceed by strong induction on m.

The thesis holds for $m = 0$ or 1 since $r^2\Delta_0 = 0$ on $\mathbb{R}[x]_m$ in these cases. For $m \geq 2$ suppose $f \in \mathbb{R}[x]_m$ satisfies $r^2\Delta_0 f = \lambda f$. By Euler's formula, $\Delta_0 f \in \mathbb{R}[x]_{m-2}$ satisfies

$$\begin{aligned}\lambda\Delta_0 f &= \Delta_0\left(r^2\Delta_0 f\right) = \Delta_0\left(r^2\right)\Delta_0 f - 4x^i\partial_i\Delta_0 f + r^2\Delta_0^2 f \\ &= -2n\Delta_0 f - 4(m-2)\Delta_0 f + r^2\Delta_0^2 f\end{aligned}$$

and hence $r^2\Delta_0(\Delta_0 f) = (\lambda+2n+4m-8)\Delta_0 f$. Therefore, either $\Delta_0 f = 0$, in which case $\lambda = 0$ and f is harmonic, or $\lambda + 2n + 4m - 8$ is an eigenvalue of $r^2\Delta_0$ on $\mathbb{R}[x]_{m-2}$ with eigenvector $\Delta_0 f$. The thesis now follows by the induction hypothesis. Indeed, since $\Delta_0 f \in \mathbb{R}[x]_{m-2}$, by induction $\Delta_0 f = r^{2j}u$ where u is harmonic. Then $f = \lambda^{-1}r^2\Delta_0 f = \lambda^{-1}r^{2(j+1)}u$ and hence $r^{2(j+1)}u$ is eigenvector on $\mathbb{R}[x]_m$ with eigenvalue $\lambda+2n+4m-8$. This is valid for all $j = 0, ..., \lfloor (m-2)/2 \rfloor$ and hence for all $j+1 = 1, ..., \lfloor m/2 \rfloor$. Because we saw pure harmonic function are eigenvector with eigenvalue 0, $r^0 u$ also works and hence the thesis, by renaming $j+1$ as j. □

Given a contravariant tensor $T \in \mathcal{T}^k(V)$, where V is a generic vector field, we denote by $\mathrm{Sym}(T)$ its symmetrisation, that is to say, given $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$,

$$\mathrm{Sym}(T)(\mathbf{v}_1, \dots, \mathbf{v}_k) = \frac{1}{k!}\sum_{\sigma \in S_k} T\left(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}\right).$$

The following theorem is an essential first step in proving our result.

Theorem 4.2. *Let $p \in M$, $k \in \mathbb{N}$ and let T be a symmetric $(k+2)$-tensor on T_pM. Then, there is a unique homogeneous polynomial $f \in \mathbb{R}[x]$ of degree $k+2$ in g-normal coordinates such that the metric $\tilde{g} = e^{2f}g$ satisfies*

$$\operatorname{Sym}\left(\nabla^k_{\tilde{g}}\tilde{R}_{ij}\right)(p) = T$$

where, as always, $\nabla_{\tilde{g}}$ denotes the connection relative to the metric $\tilde{g}$.

Proof. Let x^i as before be g-normal coordinates on a neighborhood of p, and let again $r = |x|$. Set $F_g(x) = R_{ij}(x)x^ix^j$. Then, by Taylor's theorem

$$F_g = \sum_{m=2}^{k+2} F_g^{(m)} + O\left(r^{k+3}\right),$$

where

$$F_g^{(m)} = \frac{1}{(m-2)!}\sum_{|K|=m-2} \partial_K R_{ij}(p)x^ix^jx^K \in \mathbb{R}[x]_m$$

where K are multi-indices and the sum over i and j is, as always, implied. Covariant derivatives of R_{ij} are related to ordinary partial derivatives by $R_{ij,K} = \partial_K R_{ij}(p) + S_{ijK}$, where the S_{ijK} are constructed as polynomials in the curvature and its derivatives of order $< |K|$ at p. If $\tilde{g} = e^{2f}g$ with $f \in \mathbb{R}[x]_{k+2}$, then $S_{ijK} = \tilde{S}_{ijK}$ when $|K| = k$.

Since g-normal coordinated differ from $\tilde{g}$-normal coordinates by $O(r^{k+2})$, our result is equivalent to finding $f \in \mathbb{R}[x]_{k+2}$ such that

$$\begin{aligned} 0 &= \frac{1}{k!}\sum_{|K|=k}\sum_{i,j}\left(\tilde{R}_{ij,K}(P) - T_{ijK}\right)x^ix^jx^K \\ &= F_{\tilde{g}}^{(k+2)}(x) + \sum \frac{1}{k!}\left(S_{ijK} - T_{ijK}\right)x^ix^jx^K. \end{aligned} \tag{4.1}$$

By Euler's formula $x^ix^j\partial_i\partial_j f = \left(x^i\partial_i\right)^2 f - x^i\partial_i f = (k+2)(k+1)f$. Moreover $\Delta f = \Delta_0 f + O\left(r^{k+1}\right)$. The transformation law for R_{ij} (equation 1.2) gives

$$\begin{aligned} F_{\tilde{g}}^{(k+2)}(x) &= F_g^{(k+2)}(x) + x^ix^j\left(-(n-2)\partial_i\partial_j f + \Delta_0 f\delta_{ij}\right) \\ &= F_g^{(k+2)}(x) - (n-2)(k+2)(k+1)f + r^2\Delta_0 f. \end{aligned}$$

Thus, there is a unique f so that 4.1 is satisfied provided that the differential operator $r^2\Delta_0 - (n-2)(k+2)(k+1)$ is invertible on $\mathbb{R}[x]_{k+2}$. This follows from the previous lemma. Indeed, none of the eigenvalues of $r^2\Delta_0$ on $\mathbb{R}[x]_{k+2}$ equals $(n-2)(k+2)(k+1)$, and because the eigenvalues of a matrix plus a constant change by adding the same constant it follows that no eigenvalue of $r^2\Delta_0-(n-2)(k+2)(k+1)$ is zero and therefore that the matrix is invertible. □

The following corollary is the application of the previous theorem we need for our conformal normal coordinates.

Corollary 4.3. *Given $p \in M$, $N \in \mathbb{N}$, there exists a metric conformal to g such that all symmetrized covariant derivatives of the Ricci tensor of order $< N$ vanish at p.*

Proof. By induction on N: for $N = 1$ there is nothing to prove. Now let g' by induction hypothesis be such that all symmetrized covariant derivatives of its Ricci tensor of order $< N$ vanish at p. Choose $T = 0$ in the above theorem to find g'' such that $g'' = e^{2f} g'$ and

$$\text{Sym}\left(\nabla^N_{g''} R''_{ij}\right)(p) = 0$$

and note that $f \in \mathbb{R}[x]_{N+2}$ implies that $\nabla^k_{g''} R''_{ij} = \nabla^k_{g'} R'_{ij}$ for $k < N$, which are all null by induction hypothesis. Therefore

$$\text{Sym}\left(\nabla^k_{g''} R''_{ij}\right)(p) = 0$$

for all $k \leq N$. □

We need one more technical lemma before the goal theorem, but before presenting it we are going to recall a bit of general theory on Riemannian manifolds.

Definition 4.1. A vector field along a geodesic is a *Jacobi field* if it satisfies the *Jacobi equation*

$$\frac{D^2}{dt^2} J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0$$

Where $\frac{D}{dt}$ is the covariant derivative and R is the Riemann curvature tensor.

On a complete Riemannian manifold this condition is equivalent to being a vector field J along a geodesic γ that is obtained as

$$J(t) = \left.\frac{\partial \gamma_s(t)}{\partial s}\right|_{s=0}$$

where $\gamma_s(t)$ is a one parameter family of geodesics such that $\gamma_0 = \gamma$. If the manifold is not complete such a condition is satisfied only locally.

Practically a Jacobi field aims to describe the difference between a geodesic and another geodesic infinitesimally close to it.

The next lemma shows us how to build Jacobi fields on a chart in normal coordinates which will be useful in proving the next technical lemma.

Let x^i be again g-normal coordinates on a neighborhood U of $p \in M$ and suppose the domain of such coordinates is the open and convex set Ω such that the origin is mapped to p (we can always modify a chart so to bring ourselves in such a situation). Let $\psi : \Omega \to U$ be the relative normal coordinates parametrization.

Lemma 4.4. *Let $\tau, \xi \in \Omega$ and consider the map $\gamma : (-\varepsilon, 1+\varepsilon) \times (-\varepsilon, 1+\varepsilon) \to \Omega$ given by $\gamma_s(t) = \psi(t \cdot (\tau + s\xi))$ and where $\varepsilon \in \mathbb{R}^+$ is small enough such that the image of γ is contained in Ω. γ basically gives a one parameter family of radial geodesics. The variational vector field*

$$X(\gamma_s(t)) = \psi\left(\frac{\partial}{\partial s}\gamma_s(t)\right) = \psi(t\xi)$$

is a Jacobi field, in that is satisfies the Jacobi equation.

Proof. Let $T = \gamma_s'(t)$. Recall the definition of the curvature endomorphism

$$\begin{aligned} R_T : TM &\to TM \\ X &\mapsto R(T,X)T \end{aligned}$$

Since ψ is a normal coordinate parametrisation and γ_s is the image of a family of geodesics in $\mathbb{R}^n$ it is also a family of geosdesics in M. Therefore it satisfies the geodesic equation $\nabla_T T = 0$, and because the Levi-Civita connection is by definition symmetric, $0 = \gamma_*[\partial/\partial t, \partial/\partial s] = [T, X] = \nabla_T X - \nabla_X T$. Therefore

$$\begin{aligned} 0 = \nabla_X \nabla_T T &= \nabla_T \nabla_X T - \left([\nabla_T, \nabla_X] - \nabla_{[X,T]}\right) T \\ &= \nabla_T \nabla_T X - R(T,X)T \end{aligned}$$

that is to say, X satisfies the Jacobi equation $\nabla_T^2 X = R_T(X)$. □

In the following lemma, by $\langle , \rangle_q$ we mean the inner product given by the pullback of the metric on M onto Ω, evaluated at point $q \in \Omega$.

Lemma 4.5. *Let everything be as in the previous lemma. Then*

$$\begin{aligned} \langle \xi, \xi \rangle_{t\tau} = \langle \xi, \xi \rangle_0 + \frac{t^2}{3} \langle R_\tau \xi, \xi \rangle_0 + \frac{t^3}{6} \langle (\nabla_\tau R_\tau) \xi, \xi \rangle_0 \\ + \frac{t^4}{20} \left\langle \left(\nabla_\tau^2 R_\tau\right) \xi, \xi \right\rangle_0 + \frac{2t^4}{45} \langle R_\tau \xi, R_\tau \xi \rangle_0 + O\left(t^5\right). \quad (4.2) \end{aligned}$$

Proof. Let $\tilde{\gamma} = \psi^{-1}(\gamma_0(t))$ and $\tilde{X}$ be the curve and the field on Ω, so that $\tilde{\gamma}(t) = t \cdot \tau$ and $\tilde{X}(\tilde{\gamma}(t)) = t\xi$. Define $f(t) = |X(\gamma_0(t))|^2$ (it also holds $f(t) = \left|\tilde{X}(\tilde{\gamma}(t))\right|^2$ because of the properties of normal coordinates). We now want to calculate the Taylor expansion of $f : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ in 0. We can exploit the fact that we are in $\mathbb{R}^n$ to calculate time derivatives of vectors. Moreover, we can use the fact that $\tilde{\gamma}(0) = 0$ and $\tilde{X}(\tilde{\gamma}(0)) = 0$ to calculate

$$\frac{df}{dt}(0) = \frac{d}{dt} \langle \tilde{X}(\tilde{\gamma}(t)), \tilde{X}(\tilde{\gamma}(t)) \rangle_{\tilde{\gamma}(t)} \Big|_{t=0} = 2 \langle \tilde{X}(\tilde{\gamma}(t)), \frac{d}{dt} \tilde{X}(\tilde{\gamma}(t)) \rangle_0 + 0 = 2 \langle 0, \xi \rangle_0 = 0$$

Similarly, we can calculate further derivatives (consider that from the third derivative on derivatives of the metric, and therefore the Riemannian curvature, come into play) and make simplifications thanks to the Jacobi equation above:

$$\begin{aligned} &\frac{d}{dt} f(0) = 0, && \frac{d^2}{dt^2} f(0) = 2 \langle \xi, \xi \rangle_0 \\ &\frac{d^3}{dt^3} f(0) = 0, && \frac{d^4}{dt^4} f(0) = 8 \langle R_\tau \xi, \xi \rangle_0 \\ &\frac{d^5}{dt^5} f(0) = 20 \langle (\nabla_\tau R_\tau) \xi, \xi \rangle_0 && \frac{d^6}{dt^6} f(0) = 36 \left\langle \left(\nabla_\tau^2 R_\tau\right) \xi, \xi \right\rangle_0 + 32 \langle R_\tau \xi, R_\tau \xi \rangle_0 \end{aligned}$$

Furthermore $\left|\tilde{X}(\tilde{\gamma}(t))\right|^2 = \langle t\xi, t\xi \rangle_{t\tau} = t^2 \langle \xi, \xi \rangle_{t\tau}$ and therefore by the classical Taylor expansion,

$$\begin{aligned}\langle \xi, \xi \rangle_{t\tau} =& t^{-2} \left| \tilde{X}\left(\tilde{\gamma}(t)\right) \right|^2 \\ =& \langle \xi, \xi \rangle_0 + \frac{t^2}{3} \langle R_\tau \xi, \xi \rangle_0 + \frac{t^3}{6} \langle (\nabla_\tau R_\tau) \xi, \xi \rangle_0 \\ &+ \frac{t^4}{20} \left\langle \left(\nabla_\tau^2 R_\tau\right) \xi, \xi \right\rangle_0 + \frac{2t^4}{45} \langle R_\tau \xi, R_\tau \xi \rangle_0 + O\left(t^5\right).\end{aligned}$$

□

We are now ready to prove the last lemma, which is a more precise statement of Proposition 1.14, and which will serves us as the base case of our main theorem.

Lemma 4.6. *In g-normal coordinates the function* $\det g_{ij}$ *has the expansion*

$$\begin{aligned}\det g_{ij} = 1 &- \frac{1}{3} R_{ij} x^i x^j - \frac{1}{6} R_{ij,k} x^i x^j x^k \\ &- \left(\frac{1}{20} R_{ij,kl} + \frac{1}{90} R_{pijm} R_{pklm} - \frac{1}{18} R_{ij} R_{kl} \right) x^i x^j x^k x^l + O\left(r^5\right)\end{aligned} \tag{4.3}$$

where the curvatures are evaluated at p.

Proof. Let x^i be again g-normal coordinates on a neighborhood U of p, and consider γ as in the previous lemmas. Polarizing (that is using the formula for the scalar product in terms of the norm) this with respect to ξ and by making the substitution $x = t\tau$ we get an expansion for the metric in normal coordinates

$$\begin{aligned}g_{pq}(x) =& \delta_{pq} + \frac{1}{3} R_{pijq} x^i x^j + \frac{1}{6} R_{pijq,k} x^i x^j x^k \\ &+ \left(\frac{1}{20} R_{pijq,kl} + \frac{2}{45} R_{pijm} R_{qklm} \right) x^i x^j x^k x^l + O\left(r^5\right)\end{aligned}$$

where the curvature terms are meant evaluated at the origin. This can be written

$$g_{pq} = \exp A_{pq}$$

where

$$\begin{aligned}A_{pq}(x) =& \frac{1}{3} R_{pijq} x^i x^j + \frac{1}{6} R_{pijq,k} x^i x^j x^k \\ &+ \left(\frac{1}{20} R_{pijq,kl} - \frac{1}{90} R_{pijq,k} R_{qklm} \right) x^i x^j x^k x' + O\left(r^5\right)\end{aligned}$$

then $\det g_{pq} = \exp\left(\operatorname{tr} A_{pq}\right)$ has the desired expansion. □

Finally we can prove, to a certain extent, that it is always possible to realize the condition which characterizes conformal normal coordinates.

Theorem 4.7. *Let M be a Riemannian manifold and $p \in M$. For each $N \in \mathbb{N}$, $N \geq 2$ there is a conformal metric g on M such that*

$$\det g_{ij} = 1 + O\left(r^N\right)$$

where $r = |x|$ in g-normal coordinates at p.

Proof. We are going to proceed by induction on N. The base case was given by the previous lemma. Suppose the thesis to be true for $N \geq 2$. Consider again the expansion 4.2 of $\langle \xi, \xi \rangle$ of Lemma 4.5. Each term of such expansion is of the form

$$c_k t^k \left(\left\langle \left(\nabla_\tau^{k-2} R_\tau \right) \xi, \xi \right\rangle + B_k(\xi, \xi) \right),$$

where c_k is a constant and B_k is an appropriate bilinear form constructed from R_τ and its derivatives of order less than $k-2$. Therefore, the expansion of $\det g_{ij}$ is of the form

$$\det g_{ij} = 1 + \sum_{|K|=N-2} c_N \left(R_{ij,K} - T_{ijK} \right) x^i x^j x^K + O\left(r^{N+1} \right),$$

where T_{ijK} are the coefficients of a symmetric tensor T on T_pM constructed from the curvature and its derivatives of order less than $N-2$. By Theorem 4.2 there is a unique polynomial $f \in \mathbb{R}[x]_N$ for which $\mathrm{Sym}\left(\nabla_{\tilde{g}}^{N-2} \tilde{R}_{ij} \right) = T$ for a metric $\tilde{g}$ such that $\tilde{g} = e^{2f} g$. But, by Corollary 4.3, if $f \in \mathbb{R}[x]_N$ then $T = \tilde{T}$, so $\det \tilde{g}_{ij}$ vanishes to order N in $\tilde{g}$-normal coordinates. Moreover, it vanishes to order $N+1$, because if $U \in \mathcal{T}^k$ is a rank k tensor, denoting $T(x) = T(x, ..., x)$ we have that $\mathrm{Sym}(T)(x) = T(x)$ (the proof is immediate) and therefore, in our case,

$$\begin{aligned} \sum_{|K|=N-2} c_N \left(R_{ij,K} - T_{ijK} \right) x^i x^j x^K &= \left(\nabla_{\tilde{g}}^{N-2} \tilde{R}_{ij} \right)(x) - T(x) \\ &= \mathrm{Sym}\left(\nabla_{\tilde{g}}^{N-2} \tilde{R}_{ij} \right)(x) - T(x) = 0. \end{aligned}$$

Thus the thesis is proved. □

We will call the g-normal coordinates found in the theorem *conformal normal coordinates*. The reader may accuse us of not keeping the promise of finding a metric g in the conformal class such that $\det g = 1$ exactly. However, we can reduce the error as much as we want, making it $O(r^N)$ for N arbitrarily large, which, for our purpose, is enough. We might ask ourselves if we can always find a conformal metric which achieves the desired normalization exactly in the neighborhood of p. This is actually the case, as was proven by Matthias Günther in reference [Gü93]. However, such a proof is harder than the one given and is not necessary for the solution of the Yamabe problem. Let us therefore proceed with our weaker version of conformal normal coordinates.

Corollary 4.8. *Let everything be as in the previous theorem. In conformal normal coordinates, if $N \geq 5$, the scalar curvature of g satisfies $S = O\left(r^2\right)$ and $\Delta S = \frac{1}{6}|W|^2$ at p.*

Proof. The condition $\det g_{ij} = 1$ implies that the symmetrization of the coefficients of the expansion 4.3 vanishes, so at p

$$\begin{aligned} 0 &= R_{ij} \\ 0 &= R_{ij,k} + R_{jk,i} + R_{ki,j} \\ 0 &= \mathrm{Sym}\left(R_{ij,kl} + \frac{2}{9} R_{pijm} R_{pklm} \right). \end{aligned}$$

By the first of these equalities, $R_{ijkl} = W_{ijkl}$ and

$$R_{ij,kl} - R_{ij,lk} = R^m{}_{ikl} R_{mj} + R^m{}_{jkl} R_{im} = 0.$$

By the last one

$$\begin{aligned} 0 = &\left(R_{ij,kl} + R_{kl,ij} + 2R_{ik,jl} + 2R_{jl,ik}\right) x^i x^j \\ &+ \frac{2}{9}(W_{pijm}W_{pklm} + W_{pikm}W_{pllm} + W_{pkim}W_{pjlm} \\ &+ W_{pjkm}W_{plim} + W_{pkjm}W_{plim} + W_{plkm}W_{pjim})x^i x^j. \end{aligned} \quad (4.4)$$

Now contract this last equality on the indices k and l. By making use of Proposition 1.23 and of the symmetry of the Weil tensor

$$W_{pikm}W_{pkjm} = \frac{1}{2}W_{pikm}\left(W_{pkjm} - W_{pmjk}\right) = \frac{1}{2}W_{pikm}W_{pjkm}$$

we get the following equality

$$0 = \left(3S_{,ij} + R_{ij,kk} + \frac{2}{3}W_{pikm}W_{pjkm}\right) x^i x^j.$$

Contracting on i and j we finally prove

$$\Delta S = -S_{,jj} = \frac{1}{6}|W|^2$$

at p.

Finally, by the first equality of 4.4 we get $S(p) = R_{jj}(p) = 0$ and by the second and the Bianchi identity we have $0 = \left(2R_{jk,k} + R_{kk,j}\right)(p) = 2S_{,j}(p)$, so $S = O\left(r^2\right)$. □

4.1 A partial solution to the Yamabe problem

We can now prove that the Yamabe problem has a solution as an application of conformal normal coordinates. As we anticipated, for not locally conformally flat manifolds of dimension greater than or equal to 6 we can build a local test function φ such that $Q_g(\varphi) < \Lambda$, and therefore $\lambda(M) < \Lambda$, proving the existence of a solution for the Yamabe problem. This is precisely what we are going to do in the proof of the following theorem.

Theorem 4.9 (Aubin)**.** *Let (M, g) be a compact Riemannian of dimension ≥ 6. If M is not locally conformally flat then $\lambda(M) < \lambda(S^n)$.*

Proof. Let x^i be conformal normal coordinates around $P \in M$ such that $\det g_{ij} = 1 + O(r^5)$. Let $\varepsilon \in \mathbb{R}^+$ be small enough so that the ball $B_{2\varepsilon}(0)$ is contained in the domain $\subseteq \mathbb{R}^n$ of the conformal normal coordinates. Let $\eta : \mathbb{R}^n \to [0,1]$ be, as in the proof of Theorem 3.15, a smooth radial cutoff function supported in $B_{2\varepsilon}(0)$ such that $\eta \equiv 1$ in B_ε. We are going to make use again of the functions defined in equation 3.10, which we recall for convenience ($\alpha \in \mathbb{R}^+$):

$$u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{(2-n)/2}.$$

As we saw before, in the proof of Theorem 3.15 these functions are not supported in $B_{2\varepsilon}(0)$ but $\varphi = \eta \cdot u_\alpha$ is (this is indeed the purpose of η) and therefore φ induces a well-defined function on the manifold. Since in conformal normal coordinates $dV_g = dx$, the same proof of Theorem 3.15 shows we can attain the same estimate of that proof without the factor $(1 + C\varepsilon)$, that is to say we have that

$$E(\varphi) = \int_{B_{2e}} \left(a|\nabla\varphi|^2 + S\varphi^2\right) dV_g \leq \Lambda\|\varphi\|_p^2 + C\alpha^{n-2} + C\int_{B_{2e}} S\varphi^2 dx.$$

By Theorem 4.7 we have that $S = O\left(r^2\right)$ and $\Delta S(P) = \frac{1}{6}|W(P)|^2$, so

$$\begin{aligned}\int_{B_{2e}} S\varphi^2 dx &\leqslant \int_{B_e} Su_\alpha^2 dx + C\int_{A_e} u_\alpha^2 dx \\ &= \int_0^\varepsilon \int_{S_r} \left(\frac{1}{2}S_{,ij}x^i x^j + O\left(r^3\right)\right) u_\alpha^2 d\omega_r dr + O\left(\alpha^{n-2}\right) \\ &= \int_0^\varepsilon \left(-Cr^2|W(p)|^2 + O\left(r^3\right)\right) u_\alpha^2 r^{n-1} dr + O\left(\alpha^{n-2}\right)\end{aligned}$$

and by Lemma 3.13 we have that

$$E(\varphi) \leq \begin{cases} \Lambda\|\varphi\|_p^2 - C|W(P)|^2\alpha^4 + o\left(\alpha^4\right) & \text{if } n > 6 \\ \Lambda\|\varphi\|_p^2 - C|W(P)|^2\alpha^4 \log\left(\frac{1}{\alpha}\right) + O\left(\alpha^4\right) & \text{if } n = 6 \end{cases}$$

If M is not locally conformally flat, there exists P such that $|W(P)|^2 > 0$ (recall that a manifold with $n \geq 6$ is locally conformally flat if and only if the Weil tensor vanishes everywhere), and then

$$Q_g(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2} \leq \begin{cases} \Lambda - \frac{C}{\|\varphi\|_s^2}|W(P)|^2\alpha^4 + o\left(\alpha^4\right) & \text{if } n > 6 \\ \Lambda - \frac{C}{\|\varphi\|_s^2}|W(P)|^2\alpha^4 \log\left(\frac{1}{\alpha}\right) + O\left(\alpha^4\right) & \text{if } n = 6 \end{cases} < \Lambda$$

for α sufficiently small. Thus when $n \geq 6$ $\lambda(M) < \Lambda$. □

4.2 Comments on the remaining cases

In conclusion, we will briefly talk about the structure of the proof for the remaining cases of the Yamabe problem. In such cases we aim at using the variational formulation of the problem as well, therefore our goal theorem is the following

Theorem 4.10. *If M has dimension 3, 4, or 5, or if M is locally conformally flat, then $\lambda(M) < \lambda(S^n)$ unless M is conformal to the standard sphere.*

As anticipated this theorem is indeed true and its proof is due to Schoen. Schoen's proof introduced two new important concepts. Firstly, he recognized the key role of the Green function for the operator □; in fact, his test function was simply the Green function with its singularity smoothed out. Second, he discovered the unexpected relevance of the positive mass theorem of general relativity, which had recently been proved in dimensions 3 and 4 by Schoen himself and Shing-Tung Yau (see references [SY79a], [SY79b] and [SY81]). A curious feature of Schoen's proof is that it works only in the cases not covered by Aubin's theorem. The proof of the above theorem actually requires an n-dimensional version (that at the time was yet unpublished) of the positive mass theorem, which was later announced by Schoen in reference [Sch84]. The 5-dimensional case appears to be a straightforward generalization of the 4-dimensional proof. The higher-dimensional case is more difficult. However, for $n \geq 6$ the result is needed only for locally conformally flat manifolds.

Let us examine the structure of the proof of Theorem 4.10 with a bit more detail. Even though in the previous sections we treated the cases $\lambda(M) > 0$ and $\lambda(M) \leq 0$ at the same time, in the historical development the first case has been much harder to solve, and this is still the case for the cases we are now considering. Indeed the $\lambda(M) \leq 0$ case is trivial, because being $\lambda(S^n) > 0$ it certainly holds $\lambda(M) < \lambda(S^n)$. We now consider the positive Yamabe invariant case.

In the case of the sphere we used the stereographic projection to transfer the problem on $\mathbb{R}^n$ where the analysis is much simpler. Indeed, in such a case, the pull-back of the euclidean metric $\hat{g}$ on $S^n - \{P\}$ is conformal to the standard metric $\overline{g}$ and has zero scalar curvature, which is the key property which makes the analysis on $\mathbb{R}^n$ simpler. Therefore $\hat{g} = G^{p-2}\overline{g}$ for some function G which satisfies $\Box_{\overline{g}} G = 0$. The function G is singular at P and it turns out to be a multiple of the Green function of $\Box_{\overline{g}}$ at P on S^n. Equivalently, if we start with the Green function, then $G^{p-2}\overline{g}$ defines a metric on $S^n - \{P\}$ which is isometric, through the stereographic projection, to the Euclidean metric on $\mathbb{R}^n$.

The idea is to proceed similarly for a generic compact manifold (M, g), by replacing the original metric g with the new $\Gamma_P^{p-2} g$, provided that the Green function Γ_P for $\Box_g$ exists at P. It turns out this is always the case if the Yamabe invariant is positive. We may therefore define, for such manifold, the stereographic projection.

Definition 4.2. Let (M, g) be a compact Riemannian manifold with $\lambda(M) > 0$. For $P \in M$ define the metric $\hat{g} = G^{p-2} g$ on $\hat{M} = M - \{P\}$, where

$$G = (n-2)a \operatorname{Vol}(S^n)\Gamma_P.$$

The manifold $(\hat{M}, \hat{g})$ together with the natural map $\sigma : M - \{P\} \to \hat{M}$ is called the *stereographic projection* of M from P.

$\hat{M}$ happens not to be flat, but *asymptotically flat*, in the sense of the following definition.

Definition 4.3. A Riemannian manifold (N, g) is called *asymptotically flat* of order $k \in \mathbb{N}^+$ if there exists a decomposition $N = N_0 \cup N_\infty$, with N_0 compact, and a diffeomorphism $N_\infty \to \mathbb{R}^n - B_R(0)$ for some $R \in \mathbb{R}^+$, satisfying

$$g_{ij} = \delta_{ij} + O\left(\rho^{-k}\right), \quad \partial_k g_{ij} = O\left(\rho^{-k-1}\right), \quad \partial_k \partial_l g_{ij} = O\left(\rho^{-k-2}\right)$$

as $\rho = |z| \to \infty$ in the coordinates $\{z^i\}$ induced on N_∞. The coordinates $\{z^i\}$ are called *asymptotic coordinates.*

One may think that this definition depends on the choice of asymptotic coordinates. However, it happens that the asymptotically flat structure is determined by the metric alone.

The stereographic projection may be shown to be an asymptotically flat manifold by showing the asymptotic coordinates. Let $\{x^i\}$ be conformal normal coordinates on a neighborhood U of P, and define the *inverted conformal normal coordinates* $z^i = r^{-2}x^i$ on $U - \{P\}$. It is not trivial that such coordinates are asymptotic coordinates, but indeed it turns out to be the case. As we anticipated it holds

Theorem 4.11. *The metric $\hat{g}$ is asymptotically flat of order 1 if $n = 3$, order 2 if $n \leq 4$, and order $n-2$ if M is conformally flat at P.*

The goal now is to build a test function φ on the asymptotically flat manifold $\hat{M}$ so to realize $Q_{\hat{g}}(\varphi) < \Lambda$ and to transfer the result on the original manifold M, therefore proving the sufficient condition for the problem to have a solution.

For $\alpha \in \mathbb{R}^+$ let u_α be the Sobolev extremal functions on $\mathbb{R}^n$ defined in equation 3.10. Fix a large radius $R \in \mathbb{R}^+$, let $\rho(z) = |z|$ in inverted conformal normal coordinates extended to a smooth positive function on $\hat{M}$, and let $\hat{M}_\infty = \{z \in \hat{M} \mid \rho(z) > R\}$. Define φ on $\hat{M}$ by

$$\varphi(z) = \begin{cases} u_\alpha(z) & \rho(z) \geqslant R \\ u_\alpha(R) & \rho(z) \leqslant R \end{cases}$$

with $a \gg R$ to be determined later. Observe that, as $\alpha \to \infty$, $u_\alpha(z)$ becomes very nearly constant for $|z| < R$, and so we can expect that the effect of replacing u_α by a constant inside the radius R should become negligible. Moreover, the metric on $\hat{M}$ closely approximates the Euclidean metric, and so the Yamabe quotient $Q_{\hat{g}}(\varphi)$ should become close to Λ.

Since φ is a function of the radial variable ρ alone, the behavior of $Q_{\hat{g}}(\varphi)$ as $\alpha \to \infty$ depends on the "average" behavior of the metric g over large spheres. It is useful to introduce a number $\mu \in \mathbb{R}$, which we call the *distortion coefficient* of g, that measures this average behavior. The precise definition of such a coefficient is a bit complicated, so we will not give it here, but it can be found in reference [LP87]. Its geometric meaning at infinity is analogous to that of the scalar curvature at a finite point. It is this constant that determines the values of $Q_{\hat{g}}(\varphi)$ for large α.

By using explicitly evaluating $Q_{\hat{g}}$ on the test function φ defined above, it is possible to prove

Theorem 4.12. *If $\mu > 0$, φ can be chosen so that $Q_{\hat{g}}(\varphi) < \Lambda$.*

But even supposing $\mu > 0$ we are not done yet, because the found result is valid on the manifold $(\hat{M}, \hat{g})$ and not on (M, g). However, the same proof of Theorem 3.12 shows that

$$\lambda(M) = \inf_{\psi \in C_C^\infty(\hat{M})} \frac{E(\psi)}{\|\psi\|_p^2},$$

and so approximating our test function φ by a function $\psi \in C_C^\infty(\hat{M})$, we find that $\lambda(M) < \lambda(S^n)$ if $\mu > 0$. Therefore, this results in the following theorem.

Theorem 4.13. *If (M, g) is a compact Riemannian manifold of dimension $n \geq 3$ with $\lambda(M) > 0$, then $\lambda(M) < \lambda(S^n)$ if there is a generalized stereographic projection $\hat{M}$ of M with strictly positive distortion coefficient μ.*

Such a theorem reduces the solution of the Yamabe problem in the case $\lambda(M) > 0$ to determining the sign of μ. Interestingly, calculations reveal that if $n \geq 6$ and the manifold is not locally conformally flat, the distortion coefficient may be written in terms of local geometric invariants of M (precisely at P, where the stereographic projection is taken). In the remaining cases, however, the distortion coefficient depends on a global invariant, called the *mass* of the asymptotically flat manifold $\hat{M}$.

With mass of a Riemannian manifold we mean more precisely the *ADM mass*, a concept introduced in 1960 by Arnowitt, Deser and Misner (where the acronym ADM comes from) in the context of a Hamiltonian formulation of general relativity, namely the ADM formalism.

Definition 4.4. Given an asymptotically flat Riemannian manifold (N, g) with asymptotic coordinates $\{z^i\}$, we define the *mass* as

$$m(g) = \lim_{R\to\infty} \frac{1}{\mathrm{Vol}(S^n)} \int_{S_R} \iota_{\vec{m}} dz,$$

if the limit exists, where ι is the interior product and $\vec{m}$ is the mass-density vector field defined on N_∞ by

$$\vec{m} = (\partial_i g_{ij} - \partial_j g_{ii})\, \partial_j.$$

ADM introduced this concept while studying isolated gravitational systems. They adopted the Hamiltonian viewpoint, which meant choosing a spacelike hypersurface as the initial surface and writing Einstein's equations as evolution equations from this initial data. With this approach they discovered that the mass $m(g)$ defined above is a conserved quantity and they concluded that it is the total mass of the isolated system.

Arnowitt, Deser, and Misner then conjectured that the mass, measured along a spacelike hypersurface in a physical spacetime, is nonnegative (and zero only if the spacetime is empty). Now the metric of any physical spacetime must satisfy Einstein's equation

$$R_{ij} - \frac{1}{2} S g_{ij} = T_{ij}$$

where T_{ij} is a physically reasonable energy-momentum tensor. It turns out that the energy-momentum tensors encountered in realistic physical models satisfy a certain positivity condition, called the *dominant energy condition.* By Einstein's equation this becomes a positivity condition on the Ricci tensor. In particular, for a time-independent spacetime $X = N^3 \times \mathbb{R}$ this condition is equivalent to the requirement that the scalar curvature of N (which then represents local mass density) be nonnegative. This lead to the positive mass theorem, firstly formulated in 3 dimension, but which we state directly in n dimension, that is to say the case we need.

Theorem 4.14. *Let (N, g) be an asymptotically flat Riemannian manifold of dimension $n \geq 3$ with nonnegative scalar curvature. Then $m(g) \geq 0$, with equality if and only if N is isometric to $\mathbb{R}^n$ with the euclidean metric.*

The result was proven by Schoen and Yau, and its hypothesis are actually a bit more complicated than the ones shown (but it takes too long to introduce them here). While this theorem arose in general relativity, it is a purely geometric result on asymptotically flat manifolds.

Note that in our case it is not hard to see that if the manifold M is not the sphere, $\hat{M}$ is not the euclidean space, therefore we may suppose $m(\hat{g}) > 0$.

The following proposition is the last piece of the puzzle, which completes the whole proof that the Yamabe problem always has a solution.

Proposition 4.15. *Let $\hat{M}$ be the stereographic projection of M from $P \in M$, and μ be the distortion coefficient computed with respect to inverted conformal normal coordinates. If $n < 6$ or M is conformally flat in a neighborhood of P, then $\mu = \frac{1}{2} m(\hat{g})$.*

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