UNIVERSITY OF CAMBRIDGE

Department of Applied Mathematics and Theoretical Physics

Part III essay

The Positive Mass Theorem

Supervisor: Dr. Claude WARNICK

Student:

Nicolò CAVALLERI

Year 2019-2020

Abstract

In this essay we introduce and state the positive mass theorem and then we present a rigorous version of a proof by Edward Witten. His proof makes use of some spin geometry and of some analysis on asymptotically euclidean manifolds. Therefore, a significant part of the essay is devoted to coherently introduce these two areas and expose the concepts we later use in the proof. The goal is to write down a rigorous and detailed proof, in the style of that of reference [Lee19], that can be read by any student with a good knowledge of geometry (formalizing further the work of Parker and Taubes in reference [PT82], which in turn was already a formalization of Witten's work).

Contents

1 Introduction **1**
- 1.1 Asymptotically euclidean manifolds . . . 2
- 1.2 ADM mass . . . 7
- 1.3 The positive mass theorem . . . 14

2 Spin geometry **16**
- 2.1 Clifford algebras . . . 16
- 2.2 The Spin group . . . 18
- 2.3 Spin manifolds . . . 21
- 2.4 Connections on spin manifolds . . . 23
- 2.5 The Dirac operator . . . 25

3 Asymptotically euclidean analysis **27**
- 3.1 Asymptotically euclidean analysis on spin manifolds . . . 28

4 Proof of the positive mass theorem **31**
- 4.1 The completeness hypothesis . . . 36

A Some useful global analysis **38**
- A.1 Differential operators on smooth manifolds . . . 38
- A.2 The divergence and the Laplacian . . . 40
- A.3 Sobolev spaces on Riemannian manifolds . . . 44

1 Introduction

The positive mass theorem is a well known result in mathematical general relativity. As the name suggests, it asserts that under certain hypotheses a certain quantity called "ADM mass" or more simply "mass" defined on a Riemannian manifold must be greater then or equal to zero and that it is equal to zero only if the manifold is the euclidean space.

In this introduction we define the concept of ADM mass and explain how it arises, and we also introduce and specify what are the mentioned hypothesis we make on the Riemannian manifold. This will allow us to give a precise statement of the theorem. In the next chapters we set up the environment necessary to go though one of the existing proofs of the positive mass theorem, we carry out the proof in detail and in the end, we discuss some applications. Our main references for this section and for the rest of the essay are [LP87], an excellent survey on the Yamabe problem by the American mathematicians Lee and Parker, which also includes a section on the positive mass theorem, and [Lee19], an outstanding book on geometric relativity, namely the subject which studies problems in general relativity with the tools of geometric analysis. A proof of the theorem was presented for the first time in reference [SY79], and in reference [Wit81] can be found the first sketch (chronologically) of the proof that we will go through in this essay. All concepts coming from relativity are thought with the definitions given in O'Neill's book [O'N83] which is our bible of mathematical relativity. In any case, we will try to recall the definition of most of them.

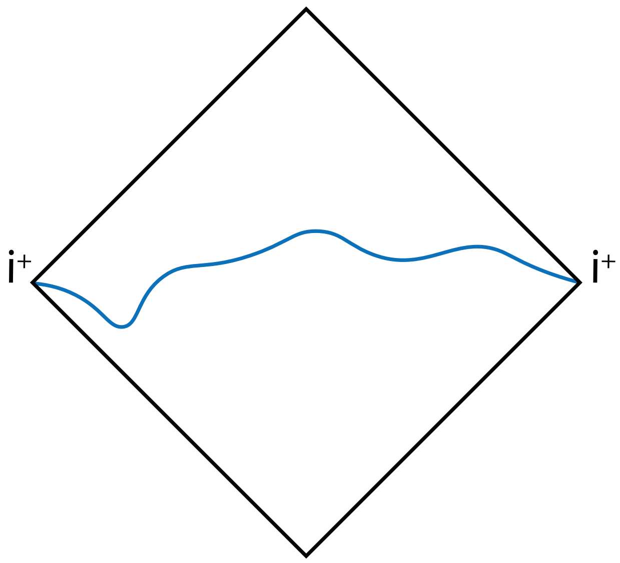

Figure 1.1: A generic Cauchy spacelike hypersurface in Minkowski space.

The concept of "ADM mass" arose for the first time in 1960, when the three American physicists Arnowitt, Deser and Misner (whose initials are indeed the origin of the acronym ADM) were trying to study in detail isolated gravitational systems (see reference [ADM60b]). To understand how they came up with this concept, suppose we have a 4-dimensional spacetime X, with Lorentzian metric g. To us a *spacetime* is a connected, time-oriented Lorentzian manifold (of arbitrary dimension). The metric satisfies Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}, \tag{1.1}$$

where, in this context, we are supposing the cosmological constant Λ to be equal to zero. To us, R_{ij} are the components of the Ricci tensor of g, R is the scalar curvature of g, g_{ij} are the tensor coordinates of the metric and T is the energy-momentum tensor. Moreover we make the assumption that there exists a "maximal" space-like hypersurface N of codimension 1, as in figure 1.1. The reason behind this is that, in their aforementioned work, Arnowitt, Deser and Misner adopted a Hamiltonian point of view, which consisted in considering a spacelike hypersurface as the "initial hypersurface" and in treating Einstein equations as evolution equations to deduce from this initial data what happens in the remaining time. To be more precise on what we mean with maximal spacelike hypersurface we neet to introduce the concept of *Cauchy surface*.

Definition 1.1. Let (X, g) be a spacetime. We say that a smooth hypersurface N (that is to say a submanifold of codimension 1) is a *Cauchy hypersurface* if every maximal (with respect to inclusion) causal curve intersects N exactly once. In case such hypersurface exists the spacetime is called *globally hyperbolic*.

We recall that a causal curve is a curve whose velocity is always causal, that is to say either timelike or null. The concept of global hyperbolicity is in some sense the analogous in spacetime geometry of completeness, which we do not want to assume in this context because some well behaving spacetimes like the classic Schwarzschild (see example 1.1), are not complete. Moreover, as the reader probably guessed, the name "Cauchy" is there to remind us that we want to think N as an initial value hypersurface. Clearly, N inherits a metric from the spacetime X, which we will call again g. The hypothesis that N is spacelike precisely means that the induced metric g on N is positive definite, and hence (N, g) is a Riemannian manifold. In the next section we want to understand what can we say about N considering that it was obtained as a Cauchy spacelike hypersurface of a spacetime with an isolated gravitational system.

1.1 Asymptotically euclidean manifolds

We begin by investigating what does "isolated gravitational system" mathematically mean and what implications does this have on the Riemannian manifold N. The physical idea is that the system is bounded to a finite region of space and it is far enough from everything else in the universe that, for the sake of studying it, we can think of it existing in an otherwise empty universe. This is translated by saying that all the mass of the universe is contained in a ball of finite radius, where the distance d is of course that induced from the metric g. Let X be a globally hyperbolic spacetime. We suppose X to be spatially infinite, that is to say for any Cauchy spacelike hypersurface N, given a point $x \in N$ there always exists a point $y \in N$ such that $d(x, y) > R$ for any given $R \in \mathbb{R}^+$. This is because we want to talk about how things look far away from the isolated system. This assumption clearly has geometrical implications on N but also topological ones, as it forces N not to be compact. A geometrical consequence of this setting is that the further we are from the isolated system the more the metric will resemble the one of a one point mass, namely the Schwarzschild metric, which in turn asymptotically looks like the flat metric. To express such a situation purely in terms of Riemannian geometry, we are naturally brought towards the definition of an asymptotically euclidean manifold

Definition 1.2. A Riemannian n-manifold (N, g), for $n \geq 2$, is said to be *asymptotically euclidean* of order $k \in \mathbb{R}^+$ if there exists $C \subseteq N$ closed and totally bounded such that $N \setminus C = \bigcup_{i=1}^{m} N_i$ where each N_i, called i-th *asymptotic end*, is an open set of N diffeomorphic to the complement of a closed disk $D_{R_i} \overset{\text{def}}{:=} \overline{B_{R_i}(0)}$ in the euclidian space, for suitable $R_i \in \mathbb{R}^+_0$, via a chart $\Phi_i : N_i \to$

$\mathbb{R}^n \setminus D_{R_i}$ satisfying the following conditions:

$$g_{ij} = \delta_{ij} + O\left(\rho^{-k}\right), \quad \partial_k g_{ij} = O\left(\rho^{-k-1}\right), \quad \partial_k \partial_l g_{ij} = O\left(\rho^{-k-2}\right)$$

as $\rho = |z| \to \infty$ in the coordinates $\{z^i\}$ induced on N_i. The coordinates $\{z^i\}$ are called *asymptotic coordinates* and the diffeomorphisms Φ_i, together with the rest of the data, make up an *asymptotic structure* on N.

Remark 1.1.

1. In the literature the most common term for this property is "asymptotically flat". However the discussion on the unit hyperboloid in example 1.2 brings up an interesting point on whether the term asymptotically flat is appropriate for the given definition. Indeed the intuition behind the term "asymptotically flat" suggests that the unit hyperboloid should have such property as it looks asymptotically like a flat manifold: the cone. Specifically, the cone (minus the singularity) is a great example of a manifold which is locally but not globally isometric to an open set of the euclidean plane, which makes it flat but not euclidean. However, the unit hyperboloid, as explained below, does not satisfy the above definition. This suggests that the most appropriate term for our definition is indeed asymptotically euclidean, which some author use and we will use as well.

2. In many typical cases which come from physics we have one asymptotic end, which we denote N_∞ and the order is $k = n - 2$. To understand why see example 1.1 below. We often denote $C = N_0$.

3. Moreover the values of R_i have clearly no topological importance, but they are necessary to realize the geometric condition $g_{ij} = \delta_{ij} + O\left(\rho^{-k}\right)$: different values may require to add a constant to the identity matrix.

4. In the above definition we would like to think C to be a compact set, but we cannot because we do not want to suppose the manifold to be complete. Indeed, many Cauchy hypersurfaces of spacetimes are not complete, and the study of such hypersurfaces is one of the main motivation to the above definition. For example, think of a Cauchy hypersurface of the classic Schwarzschild spacetime: it is not complete, but we clearly want it to be asymptotically euclidean. By classic Schwarzschild spacetime we mean the spacetime whose Penrose diagram is shown in figure 1.2, clearly not the whole Kruskal black hole. In this specific case we can get away by taking $C = \emptyset$, which is compact, but we cannot in the case of $\mathbb{R}^n \setminus K$ for any compact K containing more than 2 points, and we clearly want such manifolds to be asymptotically euclidean as well. Moreover, one of the goals of this essay is to define the ADM mass and this definition does not at all require the manifold to be complete (even though the positive mass theorem will have completeness among the hypotheses).

   A downside of this choice is that we do not have anymore a canonical bijection between asymptotic ends and topological ends. However, in the context of this essay, as we will see, the mass is defined for asymptotic ends and we will not be concerned with topological ends that are not asymptotically euclidean, hence this correspondence, even if conceptually beautiful, does not matter to us.

5. We need to use the technical property "totally bounded" instead of simply "bounded" because unfortunately, for a generic Riemannian manifold, it does not hold "bounded $\Rightarrow$ totally bounded" as one might hope. Indeed, consider the universal cover of the punctured open 2-disk with the pull-back metric through the projection. Such manifold is bounded but not totally bounded and we clearly do not want to deal with such pathologic cases. Therefore, if the manifold is complete, we can think C to be compact.

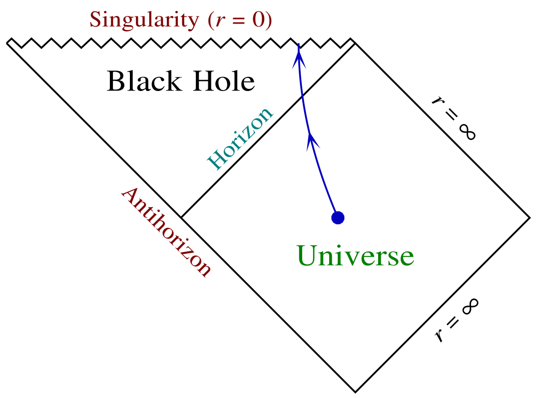

Figure 1.2: The Penrose diagram of the classic Schwarzschild black hole.

6. Note that we do not require a priori that the manifold be spatially infinite in the sense specified above, but this is a consequence of the existence of an asymptotic structure of order $k \geq 0$. Indeed let $\{z^i\}$ be asymptotic coordinates on the asymptotic end N_i. Define the curve $\gamma : [0, +\infty) \to \mathbb{R}^n \setminus D_{R_i}$ by $\gamma^1(t) = mR_i + t$ with $m = 2$ and $\gamma^i(t) = 0$ for other components. Suppose this is a geodesic for the metric $\Phi_i^*(g)$. Then

$$d(mR_i, \gamma(s)) = \int_0^s g_{ij}\dot{\gamma}^i(t)\dot{\gamma}^j(t)dt \geq \int_0^s dt - C\int_0^s \frac{1}{(mR_i + t)^k}dt$$
$$\geq s - C\int_0^s \frac{1}{t^k}dt \to +\infty \quad \text{as } s \to +\infty.$$

where the distance is that induced from the manifold. In case γ is not a geodesic we can make it arbitrarily close to a geodesic by choosing m big enough. This adds a correction term to the above calculation that can be made arbitrarily small and hence we still can find two arbitrarily distant points on N_i. This shows N_i to be spatially infinite. As discussed above, this forbids N to be compact.

7. One may think that the order in the above definition depends on the choice of asymptotic coordinates. However, it can be proved that the asymptotically euclidean structure is determined by the metric alone.

Not to leave this concept empty, we will now go through some examples of asymptotically euclidean manifolds, arising from both physics and abstract geometry.

Examples 1. 0. The zeroth example, namely the most trivial one, is clearly the euclidean space (from which we can remove a bounded set), which is asymptotically euclidean of "infinite" order with one asymptotic end. With infinite order we mean that it is asymptotically euclidean of any order.

1. The most classic example coming from physics is that of the $n+1$-dimensional classical Schwarzschild metric, where $n \geq 2$ (in physics $n = 3$). Again, we are referring to the spacetime with Penrose diagram in figure 1.2, and not to the whole Kruskal black hole. In this case, the Riemannian manifold N arises in the way described in the first part of the introduction. Topologically, the spacetime $X = \mathbb{R}^2 \times S^{n-1} \cong \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$. The metric is given by

$$ds^2 = -\left(1 - \frac{2m}{r^{n-2}}\right)dt^2 + \left(1 - \frac{2m}{r^{n-2}}\right)^{-1}dr^2 + r^2 d\Omega_{n-1}^2,$$

where polar coordinates are defined as always, $d\Omega_{n-1}^2$ ia the volume form of the $n-1$ sphere and

$$m = \frac{GM}{c^2}.$$

Here, M is the mass of the black hole, c the speed of light and G is the gravitational constant. Clearly this metric has no meaning for $r = (2m)^{\frac{1}{n-2}} := r_{EH}$, but as we are only interested to its behaviour at infinity, we do not care about what happens inside the event horizon and we use these coordinates only for $r > r_{EO}$. Therefore, even though by choosing more complicated coordinates we could deal with the whole space, we restrict our attention to $\mathbb{R} \times (\mathbb{R}^n \setminus \overline{B}_{r_{EH}})$, and we consider this to be our spacetime from now on. As a maximal spacial hypersurface we choose $N = \{0\} \times \left(\mathbb{R}^n \setminus \overline{B}_{r_{EH}}\right)$, on which the induced metric is simply

$$g = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega^2_{n-1}.$$

By asymptotically expanding the first parenthesis we can rewrite it as

$$g = dr^2 + r^2 d\Omega^2_{n-1} + \left(\frac{2m}{r^{n-2}}\right) dr^2 + O\left(r^{n-3}\right),$$

that is to say

$$g_{ij} = \delta_{ij} + O\left(r^{-(n-2)}\right),$$

that is, by taking $N_0 = \emptyset$ (which is compact), $N_\infty = N$, and $R_\infty = 0$ we see that N is an asymptotically euclidean manifold of order $n-2$. Therefore, in physics, where $n = 3$, N is of order 1. This example is the model example for any isolated gravitational system. Indeed, whatever complicated the gravitational system is, when seen from afar it will look like a point mass, and hence, through this construction, it will give rise to an asymptotically euclidean manifold of order $k = n-2$, where n are the spacial dimensions of the universe. Another typical example of isolated gravitational system is that of a binary star. In all these cases there is a unique asymptotic end N_∞. It is not hard to see, and quite intuitive, that the Kruskal black hole has two asymptotic ends, the second being identical to the first one (in particular of order $n-2$). However, we will not be concerned with such extension in this essay.

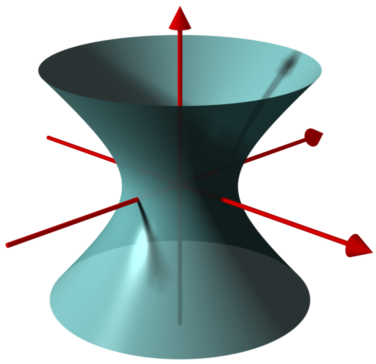

Figure 1.3: Unit hyperboloid

2. When possible, in geometry, it is always nice to have low dimensional examples that we can picture in our mind. In the case of asymptotically euclidean manifolds, a simple class of non trivial examples is given by the following surfaces in $\mathbb{R}^3$

$$H = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - h(z^2) = 1\right\}$$

where $h : [0, R) \to \mathbb{R}^+_0$ is a strictly increasing smooth function with $R \in (0, +\infty]$ and the metric is that inherited from $\mathbb{R}^3$. We also suppose that $h(0) = 0$ and that h has smooth inverse f. As our compact set C we choose the set $\{(x, y, z) \in H \mid z = 0\}$, which is closed and bounded and hence compact. The complement of such set is made up of to connected components, which are our two asymptotic ends. We call $H_+ = \{(x, y, z) \in H \mid z > 0\}$ and $H_- = \{(x, y, z) \in H \mid z < 0\}$. To see when they are asymptotic manifolds and work out the order we need to find asymptotic coordinates on each end. We will do it for H_+ as it for H_- it is absolutely analogous. As one would expect, the correct coordinates are the projections on the $z = 0$ plane. We can reconstruct z, as we can assume it is positive. Therefore the chart is

$$\mathbb{R}^2 \setminus D_1 \to H_+$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ \sqrt{f(x^2+y^2-1)} \end{bmatrix}.$$

To verify the condition on the metric and work out the order directly, we use polar coordinates on $\mathbb{R}^2 \setminus D_1$, thus getting

$$g = \left(1 + \frac{\left(f'(r^2-1)\right)^2 r^2}{f(r^2-1)}\right) dr^2 + r^2 d\theta^2,$$

and this can be written as

$$g = g_{\text{eucl}} + \frac{\left(f'(r^2-1)\right)^2 r^2}{f(r^2-1)} dr^2.$$

Hence, if $\frac{\left(f'(r^2-1)\right)^2 r^2}{f(r^2-1)} \to 0$, H is an asymptotically euclidean manifold with two asymptotic ends and with order the opposite of the exponent $\frac{\left(f'(r^2-1)\right)^2 r^2}{f(r^2-1)}$. This last term has clearly the same expansion of

$$\frac{\left(f'(r^2-1)\right)^2 (r^2-1)}{f(r^2-1)}$$

and the first term in the expansion is the square of the first term in the expansion of

$$F(x) = \frac{\left(f'(x)\right)^2 x}{f(x)}.$$

Now, if $f = \text{id}$, H is the unit hyperboloid. In this case $F(x) = 1$, hence the hyperboloid is not asymptotically euclidean. We could be more elastic with the definition and say it is asymptotically euclidean of order 0 (not a valid order according to the given definition) which is a quite intuitive geometric statement: by looking at figure 1.1 we see that the manifold asymptotically looks like a flat manifold, namely the cone, but this is not globally isometric to the plane minus a disc, which is what we really mean by asymptotically euclidean. However, we prefer to call asymptotically euclidean manifolds of order 0 "asymptotically flat".

If $h(x) = x^m$ then $f(x) = x^{\frac{1}{m}}$ and, if we understood that f gives the shape to this "modified hyperboloid" we should expect the resulting surface to be asymptotically euclidean, or at least to be closer to be asymptotically euclidean than the unit hyperboloid. In this case

$$F(x) = \frac{1}{m^2 x^{1-\frac{1}{m}}}$$

meaning that in this case the manifold is asymptotically euclidean of order $2\left(1 - \frac{1}{m}\right)$ (remember we need to multiply by two). This is interesting, as no matter what polynomial power we use to twist the hyperboloid we cannot reach more than order 2 (in fact we cannot even reach order 2). We might think that in order to reach an arbitrarily high order we could try to twist the hyperboloid so that it asymptotically tends to a plane, where here "asymptotically" is referred to the embedding of the hyperboloid in $\mathbb{R}^3$ and does not a priori have a link with the same word in the definition (though it obviously does geometrically). We can try to make the hyperboloid do so with an arbitrarily high power and see if this leads us somewhere. To do so we consider the family of functions

$$f_m(x) = \left(\frac{x}{x+1}\right)^{\frac{1}{m}}.$$

In such case simply calculating F_m would show us that the order is 6 independently from m (quite counterintuitive!). This tells us that the order behaves in a more subtle way that we might expect. In order to reach an arbitrarily high order, a working family of functions is

$$h_m(x) = -\frac{\operatorname{sgn}(y^2-1)}{|y^2-1|^{\frac{1}{m}}} - 1$$

(whose corresponding f and F can be easily calculated) which gives rise to an asymptotically euclidean manifold of order $4m+2$.

Such asymptotically flat manifolds give us precisely the right context to define ADM mass. We will do this in the next section.

1.2 ADM mass

In the very first courses of general relativity students usually learn that the Einstein equations are the Euler-Lagrange equations of the Einstein Hilbert action

$$S(g) = -\frac{c^4}{16\pi G}\int_X R_g dV_g \tag{1.2}$$

where X is the spacetime, R_g is the scalar curvature of g and dV_g is the volume form of g (we suppose the spacetime to be oriented). Since we want to work with generic manifolds, we define the dimensionless Einstein Hilbert action for an orientable pseudo-Riemannian manifold (X, g) to be

$$A(g) = -\int_X R_g dV_g, \tag{1.3}$$

so that $S(g) = \frac{c^4}{16\pi G}A(g)$. We require the manifold to be orientable so that we have a volume form, but note that we can define it on a non-orientable manifold too by using volume density. However, in the rest of this essay we will consider such action only on orientable manifolds, so we do not need to complicate things with such generalization.

One of the possible definitions of the ADM mass hides naturally in the proof that the Einstein equations are the Euler-Lagrange equations of the Einstein Hilbert action. Let us go briefly over this proof.

Lemma 1.1. *Let (X, g) be an orientable pseudo-Riemannian manifold, and let h be a symmetric 2-form on it. Let g_t be a one-parameter family of metrics such that $\left.\frac{dg_t}{dt}\right|_{t=0} = h$. Let $R_t = R_{g_t}$ and $dV_t = dV_{g_t}$ be the scalar curvature and the volume form of g_t. Then*

$$\frac{d}{dt}\left.(R_t dV_t)\right|_{t=0} = -\left(h^{jk}G_{jk} + \nabla^*\xi_h\right)dV_g \tag{1.4}$$

where G is the Einstein tensor $G_{jk} = R_{jk} - \frac{1}{2}Rg_{jk}$, ∇^* is the formal adjoint of ∇ and ξ_h is the one form

$$\xi_h = -\left(\nabla^* h + \nabla\left(\operatorname{tr}_g h\right)\right) = \left({h_{jk;}}^{k} - {h^k}_{k;j}\right)dx^j.$$

The time derivatives $\frac{d}{dt}$ are meant to be done fiberwise, where they are defined as derivatives of curves in real vector spaces. $-\nabla^*$ is the divergence, as shown in section A.2. Moreover, we use the common convention that the comma "," is associated to partial derivation and the semicolon ";" to covariant derivation.

Proof. In coordinates, the variation of the volume form $dV_t = \sqrt{|\det g_t|}dx$ is

$$\frac{d}{dt}dV_t\bigg|_{t=0} = \frac{1}{2}\sqrt{|\det g|}\,\text{sgn}(g)g^{jk}h_{jk}dx = \frac{1}{2}\,\text{sgn}(g)h^{jk}g_{jk}dV_g \tag{1.5}$$

By doing computations in normal coordinates we also get that

$$\begin{aligned}\frac{d}{dt}R_t\bigg|_{t=0} &= \frac{d}{dt}\left[g_t^{jk}\left(R_{jk}\right)_t\right]_{t=0}\\ &= -2h^{jk}R_{jk} + g^{jk}g^{il}\partial_k\left(\partial_l h_{ij} - \partial_j h_{il}\right).\end{aligned}$$

By the definition of covariant differentiation

$$h_{jk,l} = \partial_l h_{jk} - \Gamma^m_{kl}h_{jm} - \Gamma^m_{jl}h_{mk}.$$

Now let ξ_h be defined as in the statement of the lemma. We have

$$\begin{aligned}-\nabla^*\xi_h &= h_{jk,}{}^{kj} - h^k{}_{k,j}{}^j\\ &= \partial_j\partial_k h_{jk} - \partial_j\partial_j h_{kk} + \left(\partial_k\Gamma^m_{jk} - \partial_j\Gamma^m_{kk}\right)h_{jm}\\ &= \partial_j\partial_k h_{jk} - \partial_j\partial_j h_{kk} - R_{jm}h_{jm}\end{aligned}$$

where we used $R_{jm} = -R_j{}^k{}_{km}$. Thus, we get

$$\frac{d}{dt}R_t\bigg|_{t=0} = -h^{jk}R_{jk} - \nabla^*\xi_h$$

that, combined with equation 1.5, finishes the proof. □

The variation of A is thus

$$\frac{d}{dt}A\left(g_t\right)\bigg|_{t=0} = \int_X\left(h^{jk}G_{jk} + \nabla^*\xi_h\right)dV_g. \tag{1.6}$$

At this point, the original proof argues that, if we take the variation h to be compactly supported, the divergence term vanishes by the divergence theorem (which can be proved for forms the same way it can be proved for vector fields using Stokes theorem) and hence the variation of the action is

$$\frac{d}{dt}A\left(g_t\right)\bigg|_{t=0} = \int_X h^{jk}G_{jk}dV_g. \tag{1.7}$$

If we require it to be null for all compactly supported h we get that g is a critical point if it satisfies the Einstein vacuum equations $G_{ij} = 0$. In general, it is interesting to investigate what happens if we do not require h to be compactly supported, but we just require both the forms $\frac{d}{dt}\left(R_t dV_t\right)|_{t=0}$ and $R_t dV_t$ to be integrable for all ts in a neighbourhood of 0. With such hypotheses we can still apply the theorem that lets us bring the time derivative inside the integral and proceed similarly to before. However, in such case we cannot forget about the divergence term in equation 1.5. This term is pretty difficult to deal with in general, but we can significantly simplify the situation if we are working on asymptotically euclidean manifolds, and if we don't mind restricting slightly the class of h. Indeed, suppose h respects the asymptotic structure of the asymptotically euclidean n-manifold (N, g), that is to say the metric g_t is asymptotically euclidean for all ts in a neighbourhood of 0. Now let $N_1, ..., N_m$ be the asymptotic ends of N and fix asymptotically euclidean coordinates on each end. Let $D_R^i = \{x \in N_i \mid |\Phi_i(x)| \leq R\}$ and we let

$$D_R = N_0 \cup \bigcup_i D_R^i. \tag{1.8}$$

Then, by the hypothesis that $R_t dV_t$ is integrable and by considering that $R_t \chi_{D_R} \leq R_t$, by Lebesgue dominated convergence we get

$$\begin{aligned} \frac{d}{dt} A(g_t)\Big|_{t=0} &= \lim_{R\to\infty} \int_{D_R} \left(h^{jk} G_{jk} + \nabla^* \xi_h \right) dV_g \\ &= \int_N h^{jk} G_{jk} dV_g - \sum_{i=1}^m \lim_{R\to\infty} \int_{\partial D_R^i} \xi_h(\nu) dV_{\iota^*(g)} \end{aligned} \tag{1.9}$$

where ν is the normal vector to the border of D_R^i, that appears by the divergence theorem and $\iota : \partial D_R^i \hookrightarrow N$ is the inclusion. From now on, for simplicity, we will denote $dV_{\iota^*(g)}$ still with dV_g. We are now ready to give the definition of ADM mass of an asymptotic end.

Definition 1.3. Given an asymptotically euclidean manifold (N, g) we define the mass of the asymptotic end N_i to be

$$m_{\mathrm{ADM}}(N_i, g) = \lim_{R\to\infty} \frac{1}{2(n-1)\,\mathrm{Vol}(S^{n-1})} \int_{\partial D_R^i} \xi_g(\nu) dV_g. \tag{1.10}$$

When there is no confusion with other ms around we will simply write $m_i = m(N_i, g) = m_{\mathrm{ADM}}(N_i, g)$, and we will write

$$m = m(N, g) = \sum_i m_i.$$

With this definition it is immediate that

$$\frac{d}{dt}\left(A(g_t) + 2(n-1)\,\mathrm{Vol}(S^{n-1}) m(g_t) \right)\Big|_{t=0} = \int_N g^{jk} G_{jk} dV_g \tag{1.11}$$

and hence the mass is in some way a correction to the Einstein-Hilbert class, which allows us to take a larger class of variation metrics and still obtain Einstein equations. In the rest of this section we will try to give an easier expression of the mass in terms of asymptotic coordinates and try to find out why it is called mass.

In the following lemma we express the mass in the asymptotic chart.

Lemma 1.2.

$$m(N_i, g) = \lim_{R\to\infty} \frac{1}{2(n-1)\,\mathrm{Vol}(S^{n-1})} \int_{S_{R,i}^{n-1}} \bar{\xi}_g(\hat{n}) dV_{S_R^{n-1}}$$

where $\left(\bar{\xi}_g\right)_j = \left(g_{jk,}{}^{k} - g^k{}_{k,j} \right)$ *is calculated by using partial derivatives instead of covariant ones,* $\hat{n}$ *is the normal vector to* $S_{R,i}^{n-1}$ *with respect to the euclidean metric, and the subscript* i *in* $S_{R,i}^{n-1}$ *indicates that we are using the asymptotic coordinates given by the diffeomorphism* Φ_i*.*

In the definition of $\bar{\xi}_g$, it is important to note that indexes are raised and lowered through the euclidean metric and not through g. Indeed, the goal of this lemma is to put ourselves in the euclidean space and think of g as an "external" symmetric tensor rather than the metric of our space.

Proof. As the Christoffel symbols can be expressed in terms of the first derivatives of the components of the metric tensor, so, by definition of asymptotically euclidean manifold, we have that

$$(\xi_g)_j = \left(g_{jk,}{}^{k} - g^k{}_{k,j} \right) \left(1 + O\left(\rho^{-k-1} \right) \right).$$

Also the pushforward of the volume form dV_g tends to the volume form of the sphere $dV_{S_R^{n-1}}$, as much as ν tends to $\hat{n}$. Hence, by continuity of the integral

$$\lim_{R\to\infty} \frac{1}{2(n-1)} \int_{\partial D_R^i} \xi_g(\nu) dV_g = \lim_{R\to\infty} \frac{1}{2(n-1)} \int_{S_{R,i}^{n-1}} \bar{\xi}_g(\hat{n}) dV_{S_R^{n-1}}$$

□

We can further evolve the preceding result to an even simpler expression thanks to the following

Lemma 1.3. *For any 1-form ω, it holds*

$$\omega(\hat{n}) dV_{S_R^{n-1}} = \iota^* \left(\omega^\sharp \lrcorner\, d^n x \right)$$

where $d^n x$ is the standard volume form in $\mathbb{R}^n$, $\lrcorner$ the interior product and $\sharp$ the musical isomorphism inverse to $\flat$.

Proof. Let us start proving $dV_{S_R^{n-1}} = \iota^*(\hat{n} \lrcorner\, d^n x)$. For R sufficiently big, $\iota : S_R^{n-1} \hookrightarrow \mathbb{R}^n \setminus D^n$ and $\pi : \mathbb{R}^n \setminus D^n \twoheadrightarrow S_R^{n-1}$ are such that $\pi \circ \iota = \mathrm{id}_{S_R^{n-1}}$, and hence, by functoriality of the pullback $\iota^* \circ \pi^* = \mathrm{id}_{\Omega^{n-1}(S_R^{n-1})}$. Hence it is sufficient to prove $\pi^*(dV_{S_R^{n-1}}) = \hat{n} \lrcorner\, d^n x$. Indeed, by applying ι^* on both sides we would have the thesis. This, however, is straightforward, as both forms have the property of being equal to 1 on orthonormal basis in the tangent space of the sphere or radius R, and the interior product of both forms with $\hat{n}$ is null.

Now let $X = \omega^\sharp$. The thesis is equivalent to proving $\langle X, \hat{n} \rangle\, dV_{S_R^{n-1}} = \iota^*(X \lrcorner\, d^n x)$. X can be decomposed into a part parallel to $\hat{n}$, namely $\langle X, \hat{n} \rangle\, \hat{n}$, and a part orthogonal to it, which we call $X^\perp$. Thus, we have

$$\iota^*(X \lrcorner\, d^n x) = \langle X, \hat{n} \rangle\, \iota^*(\hat{n} \lrcorner\, d^n x) + \iota^*(X^\perp \lrcorner\, d^n x).$$

If we prove $\iota^*(X^\perp \lrcorner\, d^n x) = 0$ we are done by what we proved above. But this is again straightforward as $X^\perp$ can be written as a linear combination of an orthonormal basis of the tangent space of S_R^{n-1} and hence $\iota^*(X^\perp \lrcorner\, d^n x)$ evaluated on such orthonormal basis is 0. The conclusion follows from the fact that the space of n-forms over an n-dimensional vector space is one dimensional. □

Theorem 1.4.

$$m\left(N_i, g\right) = \lim_{R\to\infty} \frac{1}{2(n-1)\,\mathrm{Vol}(S^{n-1})} \int_{S_{R,i}^{n-1}} \mu \lrcorner\, d^n x \tag{1.12}$$

where μ is the vector fields defined in components by $\mu^j = \left(\partial^k g^j{}_k - \partial^j g^k{}_k \right)$.

The integral above is, as always, defined to be $\int_{S_{R,i}^{n-1}} \mu \lrcorner\, d^n x = \int_{S_{R,i}^{n-1}} \iota^*(\mu \lrcorner\, d^n x)$.

Proof. It follows immediately from the previous two lemmas. □

The expression given in the last theorem is a common definition of ADM mass in the literature.

Our next goal is to investigate whether the mass is a well-defined concept, id est we want to investigate its existence an uniqueness. By existence we mean that the integral converges and is finite for any value of $R > R_i$ and that the limit exists finite. The following result clarifies the situation completely

Proposition 1.5. *The mass of an asymptotically flat manifold exists finite if and only if the scalar curvature is integrable over the whole manifold.*

By uniqueness we mean that it does not depend on the choice of asymptotic coordinates and hence it is indeed a geometric invariant. The expression found in the last theorem, a priori, does depend on such choice. On the other hand, as the vector filed ξ_g is independent on the choice of asymptotic coordinates, we might be tempted to think that our definition does not depend on the choice of coordinates. However, if we take a closer look, the set D_R, on the border of which we are integrating, is defined through asymptotic coordinates, so a priori it also depends on the choice of coordinates. Moreover, we asserted that the two definitions are equivalent, so if one of the two does depend on the choice of coordinates so does the other. It turns out that, in general they do. An example of an asymptotically euclidean manifold of order $\frac{n-2}{2}$ whose mass depends on the choice of coordinates is given in reference [DS83]. Fortunately, however, in most of the cases in our interest we are safe, thanks to the following

Proposition 1.6. *If the order of the asymptotic manifold is strictly greater than $\frac{n-2}{2}$ the mass does not depend on the coordinates.*

In other words, which such hypotheses, the mass only depends on the metric. We will not go through the proof of this fact, which is quite long, but the original proof can be found in reference [Bar86] and a proof closer to the style of this essay can be found in reference [LP87]. In the first paper it is also shown that if the order is strictly greater than $n-2$ than the mass is 0, making these cases less interesting. Thus, we will from now on concentrate on the class of asymptotically euclidean manifolds of order in the interval $\left(\frac{n-2}{2}, n-2\right]$, for which the concept of mass is well defined and is non-trivial.

This discussion motivates the following

Definition 1.4. An asymptotically euclidean manifold (N, g) of dimension n and order k is said to admit ADM mass, to be massive or to be with mass if the mass exists finite and it is independent on the choice of coordinates, or equivalently if

1. $k > \frac{n-2}{2}$
2. $R_g \in L^1(N)$

We will end this section with some examples, where we will calculate the mass in the cases of examples 1. Hopefully this will help us to get a better idea of what the mass represents, and what geometric information in contains.

Examples 2. 0. The euclidean case is, as always, quite trivial. Indeed, by using equation 1.12, as the metric is constant in standard coordinates, the vector field μ is null and hence the mass is null.

1. The example of the Schwarzschild metric, introduced in example 1.1, is more interesting. We expect that, if ADM had some common sense when giving names, it should be equal to the mass parameter of the metric m. We recall that the metric is

$$g = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2.$$

The most handy expression to calculate the mass is that given by lemma 1.2. In order to calculate the mass with this expression, we need to calculate the components of the form

$$(\bar{\xi}_g)_j = \partial^k g_{jk} - \partial_j g^k{}_k.$$

We recall that calculations are to be done by considering the euclidean metric as the true background metric and think g as a random symmetric tensor. The trick to perform this calculation in Cartesian coordinates is to rewrite the metric as

$$g = dr^2 + r^2 d\Omega_{n-1}^2 + f(r) dr^2 = g_{\text{eucl}} + h$$

where g_{eucl} is the euclidean metric, $h = f(r)dr^2$ and

$$f(r) = \left(\left(1 - \frac{2m}{r^{n-2}}\right)^{-1} - 1\right).$$

No we note that g_{eucl} in coordinates is simply δ_{ij}, hence its partial derivative are null and we get

$$\partial^k g_{jk} - \partial_j g^k{}_k = \partial^k h_{jk} - \partial_j h^k{}_k.$$

Now we express h in Cartesian coordinates. As $dr^2 = \frac{1}{r^2} x_j x_k dx^j dx^k$

$$h_{jk} = \frac{f(r)}{r^2} x_j x_k.$$

Now simple calculations show that

$$\begin{aligned}\partial^k h_{jk} - \partial_j h^k{}_k &= x_j \frac{f'(r)}{r} + \frac{f(r)}{r^2} x_j (n-1) - x_j \frac{f'(r)}{r} \\ &= \frac{f(r)}{r^2} x_j (n-1).\end{aligned}$$

Thus

$$\begin{aligned}\bar{\xi}_g(\hat{n}) &= \frac{f(r)}{r^2} x_j \frac{x^j}{r} (n-1) \\ &= \frac{f(r)}{r}(n-1).\end{aligned}$$

Hence

$$\begin{aligned}m_{\text{ADM}}(N_\infty, g) &= \lim_{r\to\infty} \frac{1}{2(n-1)\operatorname{Vol}(S^{n-1})} \int_{S_r^{n-1}} \bar{\xi}_g(\hat{n}) dV_{S_r^{n-1}} \\ &= \lim_{r\to\infty} \frac{1}{2(n-1)\operatorname{Vol}(S^{n-1})} \int_{S_r^{n-1}} \frac{f(r)}{r}(n-1) dV_{S_r^{n-1}} \\ &= \lim_{r\to\infty} \frac{f(r)}{2r\operatorname{Vol}(S^{n-1})} \int_{S_r^{n-1}} dV_{S_r^{n-1}} \\ &= \lim_{r\to\infty} \frac{1}{2} f(r) r^{n-2}.\end{aligned} \tag{1.13}$$

This is a more general result which is valid for any spherically symmetric metric and might turn in useful later. In the specific case of the Schwarzschild metric

$$\begin{aligned}m_{\text{ADM}}(N_\infty, g) &= \lim_{r\to\infty} \frac{1}{2} r^{n-2} \left(\left(1 - \frac{2m}{r^{n-2}}\right)^{-1} - 1\right) \\ &= \lim_{r\to\infty} \frac{1}{2} r^{n-2} \frac{2m}{r^{n-2}} \\ &= m\end{aligned}$$

as we hoped. More in general, it is true that the mass of any asymptotically euclidean manifold which is a Cauchy hypersurface of a globally hyperbolic spacetime is the "total mass contained in the universe". This is the main motivation for the name mass.

2. As for example 1.2 we already know from the preceding discussion that whenever the order k is greater than zero the mass has to be 0, as $k > 0 = n - 2$. Moreover, even if to us it makes no sense the concept of asymptotically euclidean manifold of order 0, it might still be interesting to be a bit elastic with the definition and calculate the mass for the unit

hyperboloid. This because its metric takes the form of the previous example and equation 1.13 seems to produce something reasonable. Indeed, in this case

$$f(r) = \left(1 - \frac{1}{r^2}\right)^{-1}.$$

Hence, $\lim_{r\to\infty} f(r) = 1$ and equation 1.13 shows that $m(H_\pm, g) = \frac{1}{2}$. Thus $m(H, g) = \frac{1}{2} + \frac{1}{2} = 1$.

Some nice further intuitive explanation on why the ADM mass has the meaning of mass is provided in reference [Lee19]. We will try to sketch the idea. In Newtonian physics our universe is $\mathbb{R}^3$ with a mass density $\rho : \mathbb{R}^3 \to \mathbb{R}$, that can also depend on the time, but for the purpose of this discussion we care only for a frozen instant in time. Particles move according to a potential V which is determined by Poisson's equation $\Delta V = 4\pi\rho$. The total mass of the universe can be defined to be $m = \int_{\mathbb{R}^3} \rho(x)dx$. However, to define it we could also take another, less trivial point of view. The very famous Newton's shell theorem can be formulated as follow: if ρ has spherical symmetry and compact support, the potential outside of the support is equal to that of a point mass at the origin, namely $V(x) = -\frac{m}{|x|}$, where m is defined as above. Consider the case where ρ is compactly supported but does not have spherical symmetry. V is all the same harmonic outside the support of ρ, and hence, by expanding it in spherical harmonics we can recover the expression from the shell theorem by writing it as $V(x) = -\frac{m}{|x|} + O_1(|x|^{-2})$, where we say that a function $f : \mathbb{R} \to \mathbb{R}$ is $O_k(g(x))$ if $\sum_{i=0}^k x^i |f^{(i)}(x)|$ is $O(g(x))$ for $x \to +\infty$. In this expression m is just a constant that makes this approximation true; a priori it does not need to be equal to the mass defined above. However, it is a posteriori, it happens unsurprisingly that

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho(x)dx &= \int_{\mathbb{R}^3} \frac{1}{4\pi}\Delta V dx \\
&= \lim_{r\to\infty} \int_{S_r} \frac{1}{4\pi}\frac{\partial V}{\partial r} d\mu_{S_r} \\
&= \lim_{r\to\infty} \int_{S_r} \frac{1}{4\pi}\left(\frac{m}{|x|^2} + O\left(|x|^{-3}\right)\right) d\mu_{S_r}(x) \\
&= m
\end{aligned}$$

where we used the notation $d\mu$ instead of dV not to confuse the volume form with the differential of the potential. This is an approximated version of Newton shell theorem: under the sole hypothesis that the density is compactly supported the potential from afar looks like the potential of a point mass, with mass m. This mass happens to be the total mass of the universe, but this can be regarded just as a theorem: if this theorem were false, what we want this mass to be is the value which makes the approximation true, not the total mass of the universe. This is because it is this value that has physical significance: a particle located very far from where the density has its support moves according to a potential which depends from this value, and a particle close to the support does not care at all about both this value and the total mass; hence the total mass of the universe, if it did not happen to be an efficient way to calculate this parameter, is good only for bookkeeping. From the above calculation we see that a definition of the mass which reflects more how we intend it is:

$$m := \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} \frac{\partial V}{\partial r} d\mu_{S_r}.$$

At this point, $m = \int_{\mathbb{R}^3} \rho(x)dx$ is a useful theorem that follows from the linearity of the Laplacian.

Now consider an isolated gravitational system in a spacetime in general relativity, and suppose there is a complete Cauchy surface which is hence an asymptotically flat manifold (M, g). The metric here determines, through the geodesics, the trajectories of test particles and hence plays a role similar to that played by the potential V in Newtonian physics. Unlike in Newtonian physics however, the mass density $\rho : M \to \mathbb{R}$ does not determine the metric but it constrains it according to the equation $R_g = 16\pi\rho$, where R_g is the scalar curvature of g From the above discussion, it is clear that we do not want to define the mass to be $\int_M \rho dV_g$, as, again, we want the mass to tell us how test particles behave far away from the support of ρ. If the theory were linear, meaning that R_g were a linear operator of g, it would happen as in the Newtonian case that the two definitions coincide (the mass would be $\frac{1}{16\pi}\int_M R_g dV_g$). The theory unfortunately is not linear, but, taking $M = \mathbb{R}^3$, if the metric is close to the Euclidean metric globally, the scalar curvature is approximately linear, and it would be sensible to define the mass as the total integral of the linearisation of the scalar curvature $R_g \approx DR|_{\bar{g}}(g - \bar{g})$ (where $\bar{g}$ is the euclidean metric), defined to be the operator $DR|_g : \Gamma(T^*M \odot T^*M) \longrightarrow \Gamma(M)$ such that for any $\dot{g} \in \Gamma(T^*M \odot T^*M)$

$$DR|_g(\dot{g}) := \left.\frac{d}{dt}\right|_{t=0} R_{g_t}$$

for any smooth family of Riemannian metrics g_t on M such that $g_0 = g$ and $\dot{g} = \left.\frac{d}{dt}\right|_{t=0} g_t$. At this point, calculations that we do not report, show that

$$\begin{aligned} m &:= \frac{1}{16\pi}\int_{\mathbb{R}^3} DR\Big|_{\bar{g}}(g - \bar{g})dV_{Eucl} \\ &= \frac{1}{16\pi}\int_{\mathbb{R}^3}(-\bar{\Delta}(\overline{\mathrm{tr}}g) + \overline{\mathrm{div}}(\overline{\mathrm{div}}g))dV_{Eucl} \\ &= \lim_{R\to\infty}\frac{1}{16\pi}\int_{S^2_R}(\overline{\mathrm{div}}g - d(\overline{\mathrm{tr}}g))(\hat{n})dV_{S_R} \\ &= \lim_{R\to\infty}\frac{1}{4\,\mathrm{Vol}(S^2)}\int_{S^2_R}\bar{\xi}_g(\hat{n})dV_{S^{n-1}_R} \end{aligned}$$

where the bar over operators indicates that they are calculated with respect to the Euclidean background metric and not to g. Note that this is the ADM mass as expressed in 1.2. Calculations were done for a metric that is globally close to the euclidean metric, but we managed to bring ourselves in a situation where we only consider how the metric is far from where matter is, and hence it is sensible to use this expression also for metrics that are only asymptotically Euclidean for big $|x|$, that is for asymptotic ends. Hence, the deeper meaning of ADM mass is to be the necessary information to understand how test particles behave far from the origin of the chart in the considered asymptotic end.

With this interpretation, the positive mass theorem will tell us that as much as in Newtonian gravity a nonnegative mass density implies that the total mass of the universe is nonnegative, the positive mass theorem will tell us that if the scalar curvature (and hence the mass density) is nonnegative then the parameter telling us how test particles behave at big distances is positive, and hence test particles will be attracted to where matter is and not repulsed. Although, physically this statement is quite reasonable, mathematically it is far from being obvious, as the length of this essay shows.

1.3 The positive mass theorem

We are now finally ready to state the positive mass theorem.

Theorem 1.7 (Positive mass theorem)**.** *Let (N, g) be an asymptotically euclidean manifold of dimension $n \geq 3$, with nonnegative scalar curvature. Suppose further that N admits ADM mass. Then the ADM mass of each end of N is nonnegative. Moreover, if the ADM mass of any end N_i is zero, then (N, g) must be isometric to Euclidean space.*

Remark 1.2.

1. The result after the word "moreover" is also known as "positive mass rigidity" and in this context the positive mass theorem is what is stated in the first three sentences.

2. In dimension 2 the positive mass rigidity is just false. Indeed most of the twisted hyperboloids we found in example 1.2 have both ends with null mass but they are not isometric to the plane. However, by modifying the definition of asymptotically euclidean manifold we can recover a version of the theorem even in 2 dimensions. It can be found in reference [Won].

This theorem has a long history. After partial results were obtained by physicists and mathematician over a period of 20 years, Schoen and Yau finally announced a proof for the 3 dimensional case in 1979 (see reference [SY79]). Soon afterwards, they managed to generalize the proof to dimensions ≤ 8. A couple of years later, Edward Witten found a proof involving spinors (see reference [Wit81]; a more rigorous version of this proof can be found in reference [PT82] or again in reference [LP87]). Witten's proof was originally also in 3 dimensions but it can be generalized with substantially no new ideas to arbitrary dimensions. The advantage of this proof is that it is substantially simpler than the one of Schoen and Yau, which uses techniques of minimal surfaces. The drawback, however, is that this proof only applies to spin manifolds (defined later), so it is not a complete proof to the mentioned theorem. Spin manifolds, for example, are orientable so non orientable manifolds are not covered. However, we can recover the result for non orientable manifolds whose double orientable cover is a spin manifold, as discussed later.

Even if this theorem arises in general relativity, it is a purely geometric fact about asymptotically flat manifolds. The main reason why this theorem was promoted from a result in general relativity to a well-stated result in geometric analysis and extended to a generic number of dimensions, is that this theorem does have pure applications in geometric analysis. The most notable application is probably the proof of the Yamabe problem, which reads

Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, does there exist a metric conformal to g for which the scalar curvature is a constant?

The answer is yes and it is known as Yamabe's theorem. Before this theorem was proved, only a partial proof to Yamabe's theorem was known, and in order to complete that proof the positive mass theorem was needed in arbitrary dimensions. This is a notable case where a result arising in physics has an essential application to a purely geometric problem.

As mentioned at the beginning of the introduction, in the rest of this essay we will set up the machinery of spinors and we will carry out Witten's proof. We will not go though the original proof but we will carry out the more rigorous and general versions that can be found in references [LP87] and [Lee19].

2 Spin geometry

In this chapter we are going to develop all the spin geometry we need to carry out Witten's proof of the positive mass theorem. Clearly, this chapter is not meant as a proper introduction to spinors, but only to the necessary results. A quite standard reference on spin geometry for the reader who wants to delve more in to the topic is the book [LM90]. We will suppose the reader has solid knowledge in differential geometry (for example the knowledge acquired in a geometry focused part III at the university of Cambridge) and we will not go though any "standard" result in such topic.

2.1 Clifford algebras

Clifford algebras are the starting point when introducing spinors. As with many constructions in mathematics we will give the universal property and then build Clifford algebras set theoretically and show they satisfy such property. We start with defining an additional property to impose on linear maps that we later want to use in the universal property.

Definition 2.1. Let V be a vector space over a generic field $\mathbb{K}$ and let $Q : V \to \mathbb{K}$ be a quadratic form on such vector space. Moreover, let A be a unital associative algebra over $\mathbb{K}$. A linear map $\varphi : V \to A$ is said to be *Clifford-linear* if it respects the following property:

$$\forall v \in V \quad \varphi(v)^2 = -Q(v)1_A.$$

This lets us phrase conveniently the following definition.

Definition 2.2. Let (V, Q) be as above. A Clifford algebra $\mathcal{C}\ell(V, Q)$ is a unital associative algebra over $\mathbb{K}$ together with a Clifford-linear map $i : V \to \mathcal{C}\ell(V, Q)$ that respects the following universal property: for any unital associative algebra A over $\mathbb{K}$ and any Clifford-linear map $j : V \to A$ there exists a unique algebra homomorphism $f : \mathcal{C}\ell(V, Q) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\;i\;} & \mathcal{C}\ell(V,Q) \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle f} \\ & & A \end{array}$$

Note that this universal property is of the kind of the one of the tensor product. Indeed we have the poorer category of vector spaces and the richer category of associative untial algebras, and we want to build out of a vector space an algebra which has a specific property (that $\forall v(v^2 = -Q(v)1)$) that we encode by defining a specific type of linear maps (namely Clifford-linear). The property of being Clifford-linear is not easily expressible in the language of category theory, as much as the property of being bilinear in the universal property of the tensor product. It is a property that we artificially add to have an additional property in the Clifford algebra.

As it is a universal construction, it is unique up to isomorphism. To show it always exist we will build it set theoretically by defining "the" Clifford algebra.

Definition 2.3. Let (V, Q) be as above and let $\bigotimes V$ denote its tensor algebra. Moreover, let $I(V)$ denote its two sided ideal generated by the elements

$$v \otimes v + Q(v) \quad \forall v \in V.$$

We call the Clifford algebra generated by (V, Q) the algebra

$$\bigotimes V \big/ I(V)$$

Now we need to check that "the" Clifford algebra is "a" Clifford algebra, that is to say that the defined algebra satisfies the universal property for Clifford algebras.

Proposition 2.1. *The Clifford algebra defined above satisfies the universal property for Clifford algebras. Moreover the map $i : V \to C\ell(V, Q)$ is an injection.*

Proof. Let $\pi : \bigotimes V \twoheadrightarrow C\ell(V, Q)$ denote the natural projection onto the quotient and let $\iota : V \hookrightarrow \bigotimes(V, Q)$ denote the natural inclusion. Define $i = \pi \circ \iota : V \to C\ell(V, Q)$. Now let j be as in the universal property. Define $f([\otimes_k v_k]) = \Pi_k j(v_k)$ where $[]$ denotes the equivalence class. The map extends by linearity. f is well defined because j is Clifford-linear, and it respects the universal property by construction.

Now we need to check that i is injective. This, however, is quite straightforward as $i(v) = 0$ if and only if $\iota(v) \in \ker(\pi) = I$ as ι is injective. However, I does not contain pure elements of grade 1 and $\iota(v)$ is indeed a pure element of grade 1. □

We will denote the algebra product of a Clifford algebra as the normal product in $\mathbb{R}$, and we will denote $i(v)$ simply by v. $C\ell(V, Q)$ shares many similarities with the exterior algebra $\Lambda(V)$. It turns out they are isomorphic as vector spaces but not as algebras. As we will not need this fact, we shall not prove it.

One important property possessed by Clifford algebras is introduced in the following definition

Definition 2.4. An *superalgebra* (or $\mathbb{Z}_2$*-graded algebra*) is an algebra A over a commutative ring R that admits a direct sum decomposition

$$A = A_0 \oplus A_1$$

such that

$$A_i A_j \subseteq A_{i+j}$$

where the subscripts are meant to be in $\mathbb{Z}_2$. The elements of each of the A_i are called *homogeneous*. The *parity* of a homogeneous element x, denoted by $|x|$, is 0 or 1 according to whether the element is in A_0 or A_1. Elements of parity 0 are said to be *even* and those of parity 1 to be *odd*.

Note that a Clifford algebra is not, unfortunately, a graded algebra in general. As it is isomorphic to $\Lambda(V)$ as a vector space, it is a graded vector space, and we write

$$C\ell(V, Q) = \bigoplus_{k \in \mathbb{N}} C\ell^k(V, Q)$$

where $C\ell^k(V, Q)$ is defined as for exterior algebras. However, unlike the case of the exterior algebra the graded decomposition is not respected by the algebra structure. The problem is that if $Q(v) \neq 0$ we have that $v \in C\ell^1(V, Q)$ but $v \cdot v = -Q(v) \in C\ell^0(V, Q)$, whereas for $\Lambda(V)$ $v \wedge v = 0$ is in every $\Lambda^k(V)$ and hence $\Lambda(V)$ manages to be a graded algebra. However, the following lemma partially saves the situation.

Lemma 2.2. *Every Clifford algebra over a field* $\mathbb{K}$ *with* $\mathrm{char}(\mathbb{K}) \neq 2$ *is a superalgebra through the decomposition given by the operator* $\alpha : \mathcal{C}\ell(V,Q) \to \mathcal{C}\ell(V,Q)$ *which is the natural extension of the operator*

$$\begin{aligned} \alpha : V &\to V \\ v &\mapsto -v \end{aligned}$$

Proof. As $\alpha^2 = \mathrm{id}$, the operator is diagonalizable with eigenvalues 1 and -1, which are distinct as $\mathrm{char}(\mathbb{K}) \neq 2$. We define $\mathcal{C}\ell^{\mathrm{even}}(V,Q) = \ker(\alpha - \mathrm{id})$ and $\mathcal{C}\ell^{\mathrm{odd}}(V,Q) = \ker(\alpha + \mathrm{id})$. If $v, w \in \mathcal{C}\ell^{\mathrm{odd}}(V,Q)$,

$$\alpha(vw) = \alpha(v)\alpha(w) = (-1)^2 vw = vw$$

Hence $vw \in \mathcal{C}\ell^{\mathrm{even}}(V,Q)$. The other cases are analogous. □

It is not hard to verify that

$$\mathcal{C}\ell^{\mathrm{even}}(V,Q) = \bigoplus_{k\in\mathbb{N}} \mathcal{C}\ell^{2k}(V,Q).$$

2.2 The Spin group

We already know enough about Clifford algebras to introduce the spin group of a vector space. To do so we need first to talk about the pin group. We denote the group of units of $\mathcal{C}\ell(V,Q)$ with $\mathcal{C}\ell^*(V,Q)$. In such context we think of $\mathcal{C}\ell^*(V,Q)$ as a group under multiplication and we forget about the additional structure it inherits from the algebra.

Definition 2.5. Let (V,Q) be as above. The Pin group is defined to be

$$\mathrm{Pin}(V,Q) = \{v_1 v_2 ... v_k \in \mathcal{C}\ell(V,Q) \mid v_i \in V, Q(v_i) = \pm 1, k \in \mathbb{N}\}$$

with the multiplication induced from $\mathcal{C}\ell(V,Q)$. Alternatively, it is the group generated by "unit" vectors in V.

It is a group as $(v_1 v_2 ... v_k)^{-1} = \pm v_k ... v_2 v_1$. Moreover, note that $\mathcal{C}\ell^0(V,Q)$ unlike $\mathcal{C}\ell^1(V,Q)$ is a subgroup, hence it makes sense to give the following definition

Definition 2.6. Let $\mathrm{char}(\mathbb{K}) \neq 2$. The spin group of a vector space V with a quadratic form Q is defined to be

$$\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap \mathcal{C}\ell^{\mathrm{even}}(V,Q)$$

More informally $Spin(V,Q)$ is the group of products of an even number of "unit" vectors.

If $\mathrm{char}\,\mathbb{K} \neq 2$, by linear algebra we know that the theory of symmetric bilinear forms and of quadratic forms are equivalent, hence we might as well deal with inner product spaces $(V, \langle , \rangle)$, where with inner product we mean a symmetric non degenerate bilinear form (as we are working on a generic field the concept of positive definiteness makes no sense). By denoting with $|\cdot|$ the norm induced by the inner product, we write $\mathcal{C}\ell\,(V, \langle , \rangle) \overset{\mathrm{def}}{=} \mathcal{C}\ell\,\left(V, |\cdot|^2\right)$.

Lemma 2.3. *Let* $(V, \langle , \rangle)$ *be an inner product space over* $\mathbb{K}$*, and suppose* $\mathrm{char}\,\mathbb{K} \neq 2$*. With the notation defined above,*

$$\forall x, y \in V \quad x \otimes y + y \otimes x + 2\langle x, y\rangle \in I(V).$$

Proof. By definition of I

$$(x+y) \otimes (x+y) + \langle x+y, x+y\rangle \in I(V)$$

By using distributivity and using this same fact for x and y we get the thesis. □

Now let $\mathbb{K} = \mathbb{R}$. We write $\mathcal{C}\ell(p,q) = \mathcal{C}\ell(\mathbb{R}^n, \langle,\rangle)$ if $\langle,\rangle$ has signature (p,q), and we denote $\mathcal{C}\ell(n) = \mathcal{C}\ell(n,0)$ if the inner product is positive definite. Analogously we define $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}(n)$. Let $\{e_i\}$ be an orthonormal basis for $(V, \langle,\rangle)$, where $\langle,\rangle$ is positive definite. Then by the above lemma

$$e_ie_j + e_je_i = -2\delta_{ij}. \tag{2.1}$$

In case $\langle,\rangle$ is not positive definite we can recover a similar equation for an orthogonal basis:

$$e_ie_j + e_je_i = -2A_{ij} \tag{2.2}$$

where A_{ij} is a diagonal non-singular matrix.

Lemma 2.4. *$\mathcal{C}\ell(V, \langle,\rangle)$ has dimension $2^{\dim(V)}$ as a vector space.*

Proof. It is clear that $\{e_{i_1} \cdot \ldots \cdot e_{i_k} \mid 1 \leq i_j \leq \dim V\}$ generates $\mathcal{C}\ell(V, \langle,\rangle)$. Now by using equation 2.2 we see that we only need the elements for which $m < l \Rightarrow i_m < i_l$ and $k \leq n$. Counting such elements we have the thesis. □

The following theorem, which we shall not prove but whose proof can be found in reference [LM90], tells us everything we need to know about real Clifford algebras. Let $M_k(\mathbb{K})$ denote the set of $k \times k$ matrices with coefficients in $\mathbb{K}$.

Theorem 2.5. *$\mathcal{C}\ell(p,q)$ is isomorphic in the category of topological algebras to either $M_k(\mathbb{K})$ or $M_k(\mathbb{K}) \oplus M_k(\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Our next goal is to see that $\mathrm{Spin}(n)$ is a Lie group. Let $(V, \langle,\rangle)$ be again be an innner product space on a field of characteristic other than 2, let $v \in V$ and define $\rho_v \in O(V, \langle,\rangle)$ (the orthogonal group) to be

$$\begin{aligned} \rho_v &: V \to V \\ & w \mapsto w - 2\frac{\langle v, w\rangle}{\langle v, v\rangle} v \end{aligned}$$

that is, the reflection with respect to the plane orthogonal to v. Now define

$$\begin{aligned} \xi &: \mathrm{Pin}\,(V, \langle,\rangle) \to O\,(V, \langle,\rangle) \\ & v_1 \ldots v_k \mapsto \rho_{v_1} \circ \ldots \circ \rho_{v_k} \end{aligned}$$

To prove our next result we will need the classic linear algebra result that we restate.

Theorem 2.6 (Cartan-Dieudonné)**.** *Let $\langle,\rangle$ be a non-degenerate symmetric bilinear form on a finite dimensional vector space V. Then every element $g \in O(V, \langle,\rangle)$ can be written as a product of k reflections*

$$g = \rho_{v_1} \circ \ldots \circ \rho_{v_k}$$

with $k \leq \dim(V)$.

The next theorem, part of which is a corollary of Cartan-Dieudonné's theorem, is the key step in understanding the spin group. In order to prove it we will first need to prove a couple of lemmas.

Given a group G, a subgroup $H \leq G$ and a subset S we denote the centralizer in H of S by $\mathrm{Centr}_H(S) = \{h \in H \mid \forall s \in S(hs = sh)\}$.

Lemma 2.7. *If $\mathrm{char}(\mathbb{K}) \neq 2$, $\mathrm{Centr}_{\mathcal{C}\ell^*(V,\langle,\rangle)}(V) = \{\pm 1\}$.*

Proof. Let $\{e_i\}$ be an orthogonal basis of V. For an element $x \in \mathcal{C}\ell^*(V, \langle,\rangle)$ commuting with every element of V is equivalent to commuting with e_i for all i. A basis for $\mathcal{C}\ell(V, \langle,\rangle)$ is given by the elements of the form $e_{i_1} \cdot \ldots \cdot e_{i_k}$ with $i_l < i_m$ if $l < m$, as seen in the proof of lemma 2.4. Let x be decomposed as a sum in this basis. Now take the addend with the highest number of

e_i in the product, or one of them in case there is multiples. If its number of factors is odd, its coefficient changes sign when it is multiplied by any e_i on different sides. If it is even and greater then 0, multiplying it by one of the e_i in the product on different sides, causes the coefficient of the term resulting from such multiplication to be different depending on the side on which we take the product (because $\text{char}(\mathbb{K}) \neq 2$). Hence the only possibility is that x is an element of the field. In this case it has clearly to be $x^2 = 1$ and since $\text{char}(\mathbb{K}) \neq 2$, $x = \pm 1$. □

It follows that $\text{Centr}_{\text{Pin}(V,\langle,\rangle)}(V) = \text{Centr}_{\text{Spin}(V,\langle,\rangle)}(V) = \{\pm 1\}$.

The other lemma that we need concerns the adjoint representation

$$\text{Ad} : \mathcal{C}\ell^*(V, Q) \to \text{Aut}(\mathcal{C}\ell(V, Q))$$

defined by

$$\text{Ad}_\varphi(x) = \varphi x \varphi^{-1}.$$

Lemma 2.8. *Let $v \in V \subset \mathcal{C}\ell(V, \langle,\rangle)$ be such that $\langle v, v\rangle \neq 0$. Then $\forall w \in V$ $\text{Ad}_v(w) = -\rho_v(w)$.*

Proof. It is clear that $v^{-1} = -\frac{v}{\|v\|^2}$. Hence, by lemma 2.3

$$\begin{aligned} -\|v\|^2 \text{Ad}_v w &= -\|v\|^2 vwv^{-1} = vwv = w\|v\|^2 + vwv + wvv \\ &= w\|v\|^2 + (vw + wv)v = w\|v\|^2 - 2\langle v, w\rangle v \end{aligned}$$

□

Theorem 2.9. *The map ξ is surjective and its kernel is $\{\pm 1\}$. Moreover $\xi(\text{Spin}(V, \langle,\rangle)) = SO(V, \langle,\rangle)$.*

Partial proof. The surjectivity follows directly from Cartan-Dieudonné's theorem. We will work out the kernel only for the restricted map $\xi : \text{Spin}(V, \langle,\rangle) \to SO(V, \langle,\rangle)$. Firstly, let us prove that the map does indeed restrict to a map $\xi : \text{Spin}(V, \langle,\rangle) \to SO(V, \langle,\rangle)$. We have that

$$\det(\xi(s)) = (-1)^{2k} = 1 \quad \forall s \in \text{Spin}(V, \langle,\rangle)$$

hence $\xi(\text{Spin}(V, \langle,\rangle)) \subseteq SO(V, \langle,\rangle)$. However, the other inclusion has to be true as well because $\det(\xi(s)) = -1$ whenever $s \in \text{Pin}(V, \langle,\rangle) \setminus \text{Spin}(V, \langle,\rangle)$ and hence if the restricted ξ were not surjective, the original ξ could not be surjective either. To calculate the kernel, we note that $\xi(s)(x) = sxs^{-1}$. Indeed, let $s = v_1 \cdot ... \cdot v_{2k}$, with v_i unit vectors. Then, by lemma 2.8,

$$\xi(s)(x) = \rho_{v_1} \circ ... \circ \rho_{v_{2k}}(x) = (-1)^{2k} \text{Ad}_{v_1} \circ ... \circ \text{Ad}_{v_{2k}}(x) = \text{Ad}_s(x) = sxs^{-1}.$$

Now $\forall x \in V(sxs^{-1} = x)$ is equivalent to $\forall x \in V(sx = xs)$. Hence, by lemma 2.7 we have the thesis. □

Calculating the kernel for the general ξ is harder and the complete proof of the theorem can be found in reference [LM90]. This proof does not work in such case because in working out $\xi(s)(x)$ we get $(-1)^{2k+1}$ instead of $(-1)^{2k}$.

The next theorem is probably the main reason for the importance of the spin group.

Theorem 2.10. *Let $n \geq 3$. Then $\text{Spin}(p, q)$ is a connected Lie group which double covers $SO(p, q)$.*

Partial proof. To see that it is a Lie group note that $\mathcal{C}\ell(p, q)$ is diffeomorphic to $\mathbb{R}^n$ as a smooth manifold. $\mathcal{C}\ell^*(p, q)$ is an open subset and hence it is also a smooth manifold. To see it is a open subset recall that by theorem 2.5 $\mathcal{C}\ell^*(p, q)$ is isomorphic to either $M_k(\mathbb{K})$ or $M_k(\mathbb{K}) \oplus M_k(\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In case it is diffeomorphic to $M_k(\mathbb{K})$ the statement is well known. In case it is diffeomorphic to $M_k(\mathbb{K}) \oplus M_k(\mathbb{K})$ we can prove it similarly to the case of $M_k(\mathbb{K})$. Indeed, in such case it is immediate that an element is invertible if and only if both its components are invertible. Hence, the determinant of both its component must be nonzero and hence the product of the determinants must be nonzero. Thus, define the continuous function

$$\begin{aligned} d : M_k(\mathbb{K}) \oplus M_k(\mathbb{K}) &\to \mathbb{K} \\ (A, B) &\mapsto \det(A)\det(B) \end{aligned}$$

$d^{-1}(0)$ is closed and hence the set of invertible elements is open. Now, as $\xi : \text{Spin}(p, q) \to SO(p, q)$ is continuous and surjective $\text{Spin}(p, q)$ is compact and hence closed (indeed $\mathcal{C}\ell^*(p, q)$ is Hausdorff, being homeomorphic to an open subset of $\mathbb{R}^n$). Thus, by the closed subgroup theorem, $\text{Spin}(p, q)$ is itself a Lie group.

To see it is connected it suffices to show that 1 and -1 are path connected. Indeed being connected for a manifold is equivalent to be path connected, and to show it is path connected it is sufficient to show that every element is connected to 1. Now take an element $x \in \text{Spin}(p, q)$ and take the path joining in $SO(p, q)$ the identity to $\xi(x)$. Lifting this path either from 1 or -1 will end up in x. In case it happens starting from 1 we are done, otherwise, supposing we built a path from 1 to -1 we can just compose it with this path. Hence, let us build such path. As $n \geq 3$, either p or q will be ≥ 2. Suppose p is such (the case of q is analogous). Then there exist two orthogonal element v, w of norm 1. The path

$$\begin{aligned} \varphi : [0, 1] &\to \text{Spin}(p, q) \\ t &\mapsto -v \cdot (\cos(\pi t)v + \sin(\pi t)w) \end{aligned}$$

works.

From differential geometry we have that ξ is a covering map if and only if $d\xi_1 : T_1 \text{Spin}(p, q) \to T_{\text{id}}SO(p, q)$ is invertible. This calculation can be partly found in reference [Lee19]. □

The following corollary is also of primary importance.

Corollary 2.11. *For $n \geq 3$ $\text{Spin}(n)$ is simply connected and hence it is the universal cover of $SO(n)$. In particular, $\dim \text{Spin}(n) = \dim SO(n) = \frac{n(n-1)}{2}$.*

Proof. It is a well known fact that $\pi_1(SO(n)) = \mathbb{Z}_2$ when $n \geq 3$. Hence, as $\text{Spin}(n)$ is connected, it has to be the universal cover by the theory of covering spaces. Moreover, it is straightforward that a covering Lie group has the same dimension of the base group. □

Example 1. In the very first courses in differential geometry students learn that $SU(2)$ is the universal cover of $SO(3)$. Hence, by uniqueness of the universal cover, $\text{Spin}(3) \cong SU(2)$ in the category of Lie groups.

2.3 Spin manifolds

We are now ready to define the class of manifolds where we will prove the positive mass theorem. We recall that the oriented orthonormal frame bundle $F_{SO}(M)$ of an oriented Riemannian manifold, is just the frame bundle associated with its tangent bundle.

Definition 2.7. Let (M, g) be an oriented Riemannian n-manifold, and let $\pi_{SO} : F_{SO}(M) \to M$ be its oriented orthonormal frame bundle. A *spin structure* on M is a pair (π_{Spin}, E) where

1. $\pi_{\text{Spin}} : F_{\text{Spin}} \to M$ is a principal $\text{Spin}(n)$-bundle over M
2. $E : F_{\text{Spin}} \to F_{SO}(M)$ is a two-fold covering map such that

   $$\pi_{SO} \circ E = \pi_{\text{Spin}} \text{ and } E(pq) = E(p)\xi(q) \text{ for all } p \in F_{\text{Spin}} \text{ and } q \in \text{Spin}(n)$$

   where ξ is the map defined above. An oriented Riemannian manifold is said to be a *spin manifold* if it admits a spin structure.

Remark 2.1. 1. If X, Y are G-sets, where G is a group, a map $f : X \to Y$ is said to be *equivariant* if $\forall g \in G\ \forall x \in X\ (f(g \cdot x) = g \cdot f(x))$. Spin$(n)$ has a natural right action on itself by right multiplication and a natural right action on $SO(n)$ by right multiplication previous composition with ξ, hence the property $E(pq) = E(p)\xi(q)$ may be summarized by "equivariancy on the fibres" or simply equivariancy.

2. The equation $\pi_{SO} \circ E = \pi_{\text{Spin}}$ is saying that the map π_{Spin} lifts to E with respect to the fibre bundle π_{SO}. Together with the previous remark this lets us reformulate the definition as: a *spin structure* on M is a principal Spin(n)-bundle over M that lifts to an equivariant two-fold covering map with respect to the oriented orthonormal frame bundle.

3. It turns out that the existence of a Spin structure is equivalent to the vanishing of a specific characteristic class, the second Stiefel-Whitney class. This tells us that being a spin manifold is a purely topological property, independent on the metric. Hence, from now on we will ask ourselves whether a manifold is spin or not and not if a Riemannian manifold is.

4. The intuitive idea of the above definition is to try to reproduce fibrewise the double cover of Spin(n) on $SO(n)$ and to do so in such a way that it makes sense globally. Indeed, when working with transition functions, we can always lift the $SO(n)$-valued transition functions for F_{SO} locally to $Spin(n)$-valued transition functions in two different ways. The property for a manifold of being spin means that this can be done in such a way that the cocycle conditions are satisfied, yielding a global construction.

We are now able to define spinors.

Definition 2.8. A *spinor bundle* on a smooth manifold with a given spin structure is a vector bundle ρ-associated to the spin group principal bundle π_{Spin} given by the spin structure, where $\rho : \text{Spin}(n) \to GL(V)$ is a representation of the spin group. Section of such vector bundles are called *spinor fields* or simply *spinors*.

Remark 2.2. If the representation ρ descends to a representation over $SO(n)$, id est if there exists a map $\tilde{\rho} : SO(n) \to GL(V)$ such that $\rho = \tilde{\rho} \circ \xi$, then we get nothing new than a vector bundle with group structure $SO(n)$. However, if such map does not exist we get new interesting objects, which behave differently than normal vectors. Even if the following statement is very unprecise, we might think intuitively that when rotating by 2π a vector in a fibre space of a vector bundle with structure group $SO(n)$ we get the same vector back, when we rotate it by 2π in a fibre space of a vector bundle with structure group Spin(n) (such that it does not descend to $SO(n)$) we get its opposite. The very interesting side of this is that some physical quantities display such behaviour.

There is a specific class of spinor bundles that has a particular interest, which is indeed that of spinor bundles which admit a Clifford action. In order to define what such action is we need to firstly define Clifford bundles.

Definition 2.9. A Clifford bundle is an algebra bundle whose typical fibre is a Clifford algebra. Algebra bundle means that the transition functions of the local trivializations are fibrewise algebra isomorphisms.

There is one Clifford bundle which is more important than the others.

Definition 2.10. Given an oriented Riemannian manifold (M, g) we define the *canonical Clifford bundle* $\mathcal{C}\ell(M)$ to be the Clifford bundle associated to the oriented orthonormal frame bundle $F_{SO}(M)$ through the representation $\rho : SO(n) \to GL(\mathcal{C}\ell(n))$ given by

$$\forall g \in SO(n)\ \forall v_1, ..., v_k \in \mathbb{R}^n \quad \rho(g)(v_1 \cdot ... \cdot v_n) = g(v_1) \cdot ... \cdot g(v_k).$$

Note that for this construction only the metric is needed and not a spin structure.

Given S a vector space that has the structure of a real module over $\mathcal{C}\ell(n)$ we can build a spinor bundle in a canonical way, on which the canonical Clifford bundle acts naturally.

Since every element of $\mathrm{Spin}(n)$ is a Clifford product of unit vectors, $\mathrm{Spin}(n)$ acts on S, yielding a representation. Now we can use such representation and the spin structure to build the spinor bundle associated with the spin bundle F_{Spin} to get a spinor bundle that we will denote $S(M)$. The canonical Clifford bundle has a natural action fibrewise, induced by the module structure of S over $\mathcal{C}\ell(n)$, which we will call the Clifford action and which we will still denote with $\cdot$.

Moreover $S(M)$ is canonically a Riemannian bundle, as the following proposition shows

Proposition 2.12. *$S(M)$ carries a metric such that unit vectors $v \in \mathbb{R}^n \subset \mathcal{C}\ell(n)$ act on $S(M)$ orthogonally, meaning that for all $\varphi, \psi \in \Gamma(S(M))$*

$$\langle v \cdot \varphi, v \cdot \psi \rangle = \langle \varphi, \psi \rangle$$

where $\cdot$ denotes the action of $\mathcal{C}\ell(n)$ induced by the module structure.

The proof can be found in reference [LM90] (remark III.10.2 and proposition I.5.16). Now, as $v^2 = -1$ it follows that v acts as a skew symmetric operator on $S(M)$. Indeed

$$\langle v \cdot \varphi, \psi \rangle = \langle v \cdot v \cdot \varphi, v \cdot \psi \rangle = -\langle \varphi, v \cdot \psi \rangle.$$

2.4 Connections on spin manifolds

The Levi-Civita connection over M extends to a connection on both $\mathcal{C}\ell(M)$ and $S(M)$. On $\mathcal{C}\ell(M)$ it is extended this way: being M a Riemannian manifold, the Levi-Civita connection induces a connection on the oriented orthonormal frame bundle $F_{SO}(M)$ (in a quite obvious way as it is clear how to parallel transport frames) and this induces a connection on any induced vector bundle, whatever the representation is. This second construction is not trivial but all the same quite standard in differential geometry. A good reference where to find it is section III.11.8 of reference [Kol93]. Although it is not obvious, the connection obtained this way obeys the Leibniz rule on the algebra structure $\nabla(\sigma\tau) = (\nabla\sigma)\tau + \sigma(\nabla\tau)\ \forall\sigma, \tau \in C^\infty(\mathrm{Cl}(M))$.

For $S(M)$ it is a bit more complicated. The most obvious thing to do is to build a connection on F_{Spin} and then transfer it to $S(M)$ through the construction we mentioned above (that, being standard, we do not go through in this essay). The problem though is that even for F_{Spin} there is not a standard construction in elementary differential geometry to do it. However, being it a covering space of a bundle where we have a standard connection, namely $F_{SO}(M)$ we should be able to somehow lift a connection on it. Specifically, as one of the central features of covering spaces is that paths always lift in a unique way, a good idea might be to lift parallel transport. Let us do this explicitly: even if it is clear what the only thing that can be done here is, it takes quite long to explain it. Let $a \in F_{\mathrm{Spin}}$ and let $p = \pi_{\mathrm{Spin}}(a)$. Let $U \ni a$ be a trivializing neighbourhood for $F_{SO}(M)$ and, by restricting it if necessary, assume also that $Y = \pi_{SO}^{-1}(U)$ a trivializing neighbourhood for the covering map $F_{\mathrm{Spin}} \twoheadrightarrow F_{SO}(M)$. The fact that it is trivializing for F_{SO} means there is a global section on U that we denote with $e = (e_1, ..., e_n)$. By rotating the section with a constant element of $SO(n)$ if necessary we can choose $e(p) = E(a)$. Choosing the preimage $\tilde{Y} \subseteq F_{\mathrm{Spin}}$ of Y such that $a \in \tilde{Y}$, this section clearly lifts to a section $\tilde{e}$ on $\tilde{Y} = \pi_{\mathrm{Spin}}^{-1}(U)$ ($E : \tilde{Y} \to Y$ is a diffeomorphism) such that $\tilde{e}(p) = a$. Now let $\gamma : I \to U$ be a path with $\gamma(0) = s$ and consider parallel translation along γ. Then, because local trivialisations are diffeomorphisms, there exist $g : I \to SO(n)$ such that $g(0) = \mathrm{id}$ and $t \mapsto (\gamma(t), g(t)) \in U \times SO(n)$ is the image through the local trivialisation of the parallel transport of $e(p)$ along γ in F_{SO}. Now, consider the lift $\tilde{g}$ of g from $SO(n)$ to $\mathrm{Spin}(n)$ with $\tilde{g}(0) = 1$. Then we define $t \mapsto (\gamma(t), \tilde{g}(t))$ (more precisely its preimage through the trivialisation) to be the parallel transport of a along γ in F_{Spin}. The connection associated to this parallel transport is the desired connection, and as we said this induces a connection on $S(M)$.

The goal now is to find an explicit expression for covariant differentiation on $S(M)$. The following definition will reveal to be interesting after we will have used it in calculations.

Definition 2.11. A local section of $S(M)$ (namely a local spinor) defined on a trivialising neighbourhood U of $F_{SO}(M)$ is said to be *constant* with respect to the frame $e = (e_1, ..., e_n)$ if it is constant with respect to the local trivialisation associated to the local trivialisation generated by the section on U in the case of $F_{SO}(M)$ (recall that in constructing a vector bundle from a principal bundle there is a correspondence between the trivialisations).

Now, let $\{e^i\}$ be the dual frame to $\{e_i\}$. Moreover, let $\Xi = d\xi_1$ where

$$d\xi_1 : \mathfrak{spin}(n) \stackrel{\text{def}}{:=} T_1 \operatorname{Spin}(n) \longrightarrow \mathfrak{so}(n) \stackrel{\text{def}}{:=} T_{\text{Id}}\text{SO}(n)$$

is the map induced by ξ on the Lie algebras. Note that as much as $\mathfrak{so}(n)$ can be regarded as a subspace of $\operatorname{End}(\mathbb{R}^n)$, $\mathfrak{spin}(n)$ is identified with a subspace of $\mathcal{C}\ell(n)$. From the previous discussion it is clear that $\mathfrak{spin}(n)$ is isomorphic to $\mathfrak{so}(n)$, but we still will denote it with $\mathfrak{spin}(n)$ when we want to think of it as a subspace of $\mathcal{C}\ell(n)$. The proof of the following result is not hard but a bit too long for what this section is intended to be.

Lemma 2.13. *Let e_i be the standard basis of $\mathbb{R}^n$. Let*

$$A_i^j := e_j \otimes e^i - e_i \otimes e^j \in \operatorname{End}(\mathbb{R}^n).$$

Then

$$\Xi(e_i e_j) = 2A_j^i.$$

Now we write everything in a local trivialisation. In particular, consider a local trivialisation of the tangent bundle of M $\pi^{-1}(U) \to U \times \mathbb{R}^n$. Let $\gamma'(0) = X$. Then, in p, $\forall v \in \mathbb{R}^n$ we get

$$\begin{aligned} 0 &= \nabla_X(g(t)v) \\ &= g'(0)v + \nabla_X v \\ &= g'(0)v + \omega_j^i(X)\left(e_i \otimes e^j\right) v \end{aligned}$$

where ω_j^i are the connection 1-forms determined by the local trivialisation. We get to the second line from the first by the relation between parallel transport on the frame bundle and covariant differentiation on the original line (further justification should be provided but calculations are too long). The third line follows from the second by the fact that covariant derivative is expressed in terms of the connections one forms by $\nabla_X = d + \omega(X)$, and then we expand out $\omega(X)$ on the basis $e_i \otimes e^j$.

Now we use the fact that the matrices ω are antisymmetric for principal bundles together with lemma 2.13 to write

$$g'(0) = -\omega_j^i(X) e_i \otimes e^j = -\frac{1}{2}\omega_j^i(X) A_i^j = -\frac{1}{4}\sum_{i,j} \omega_j^i(X)\Xi(e_i e_j).$$

Let $s \in S$. It turns out, that in the local trivialisation the parallel transport of s is $\tilde{g}(t)s$. Again, by using without proof results on the relation between parallel transport and covariant derivatives, we write

$$0 = \nabla_X(\tilde{g}(t) \cdot s) = \tilde{g}'(0) \cdot s + \nabla_X s$$

and hence, with the expression for $\tilde{g}'(0)$ found above,

$$\nabla_X s = \frac{1}{4}\sum_{i,j=1}^{n} \omega_j^i(X) e_i e_j \cdot s.$$

All of this is summed up by saying that if a spinor $\psi \in \Gamma(S(M))$ is constant with respect to a given frame, then

$$\nabla\psi = \frac{1}{4}\sum_{i,j=1}^{n} \omega^i_j e_i e_j \cdot \psi. \tag{2.3}$$

As we might expect we have the following result

Proposition 2.14. *The connection obtained on $S(M)$ is a metric connection, with respect to the metric obtained in proposition 2.12. This means that for all spinors $\phi, \psi \in \Gamma(S(M))$ and for all vectors $X \in \mathfrak{X}(M)$ it holds*

$$X\langle\phi,\psi\rangle = \langle\nabla_X\phi,\psi\rangle + \langle\phi,\nabla_X\psi\rangle.$$

Proof. Let us work in a trivialisation and let $s_1, ..., s_m$ be a system of constant orthonormal spinors for the trivialisation. Then

$$\phi = \phi^i s_i \qquad \psi = \psi^i s_i$$

where the coordinates are just smooth functions. Then, by making use of equation 2.3,

$$\begin{aligned}
\langle\nabla_X\phi,\psi\rangle + \langle\phi,\nabla_X\psi\rangle &= X(\phi^i)\langle s_i,\psi\rangle + X(\psi^j)\langle\phi,s_j\rangle + \phi^k\langle\nabla_X s_k,\psi\rangle + \psi^\ell\langle\phi,\nabla_X s_\ell\rangle \\
&= \psi^j X(\phi^i)\delta_{ij} + \phi^i X(\psi^j)\delta_{ij} \\
&\quad + \frac{1}{4}\left(\phi^k\left\langle\sum_{i,j=1}^{n}\omega^i_j(X)e_ie_j\cdot s_k,\psi\right\rangle + \psi^\ell\left\langle\phi,\sum_{i,j=1}^{n}\omega^i_j(X)e_ie_j\cdot s_\ell\right\rangle\right) \\
&= X(\psi^j\phi^i\delta_{ij}) - \frac{1}{4}\sum_{i,j=1}^{n}\omega^i_j(X)\left(\phi^k\langle e_j\cdot\phi, e_i\cdot\psi\rangle + \psi^\ell\langle e_i\cdot\phi, e_j\cdot s_\ell\rangle\right) \\
&= X(\langle\phi,\psi\rangle) - \frac{1}{4}\phi^k\psi^\ell\sum_{i,j=1}^{n}\omega^i_j(X)\left(\langle e_j\cdot\phi, e_i\cdot s_\ell\rangle + \psi^\ell\langle e_i\cdot s_k, e_j\cdot s_\ell\rangle\right) \\
&= X\langle\phi,\psi\rangle
\end{aligned}$$

where in the last equation we used again the skew-symmetry of ω^i_j. □

The proof of the following proposition would require developing further the theory of connections on spin bundles, so we will not go through it, but we will need all the same the result.

Proposition 2.15. *The connection respects the module structure of $S(M)$ over $\mathcal{C}\ell(M)$, that is to say, for all $v \in \Gamma(TM)$, $\psi \in \Gamma(S(M))$*

$$\nabla(v\cdot\psi) = (\nabla v)\cdot\psi + v\cdot(\nabla\psi).$$

2.5 The Dirac operator

In this section we will define a differential operator which plays a central role in the proof of the positive mass theorem, the Dirac operator. Mathematicians call Dirac operator whatever operator is a formal square root of the Laplacian. In this context, it refers to a specific operator.

Definition 2.12. The Dirac operator is the operator $\not{D} : \Gamma(S(M)) \longrightarrow \Gamma(S(M))$ defined by

$$\not{D}\psi = \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}\psi$$

where $e_1, ..., e_n$ is a local orthonormal frame and $\cdot$ denotes the Clifford module multiplication. The spinors that satisfy the equation $\not{D}\psi = 0$ are said to be *harmonic spinors*.

Proposition 2.16. *The Dirac operator is well defined, meaning that the above definition is independent of choice of local orthonormal frame.*

Proof. Suppose $\tilde{e}_1, ..., \tilde{e}_n$ is another orthornormal frame. Then there exist A_i^j smooth such that $e_i = A_i^j \tilde{e}_j$ and $A \in SO(n)$ at every point. Hence

$$\sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \psi = \sum_{i=1}^{n} A_i^j \tilde{e}_j \cdot \nabla_{A_i^k \tilde{e}_k} \psi = \sum_{i=1}^{n} A_i^j A_i^k \tilde{e}_j \cdot \nabla_{\tilde{e}_k} \psi = \delta^{jk} \tilde{e}_j \cdot \nabla_{\tilde{e}_k} \psi = \sum_{i=1}^{n} \tilde{e}_i \cdot \nabla_{\tilde{e}_i} \psi.$$

□

Also formal self adjointness is a straightforward verification (and a key property for the development of this essay).

Lemma 2.17. *$\not{D}$ is self adjoint, meaning that if either one of $\varphi, \psi \in \Gamma(S(M))$ is compactly supported,*

$$\langle\langle \not{D}\varphi, \psi \rangle\rangle = \langle\langle \varphi, \not{D}\psi \rangle\rangle.$$

Proof. Choose at every point a local orthonormal frame parallel at the given point. Then compute

$$\begin{aligned}
\langle \not{D}\varphi, \psi \rangle &= \left\langle \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \varphi, \psi \right\rangle = -\left\langle \nabla_{e_i} \varphi, \sum_{i=1}^{n} e_i \cdot \psi \right\rangle \\
&= \left\langle \varphi, \sum_{i=1}^{n} \nabla_{e_i} (e_i) \cdot \psi \right\rangle + \left\langle \varphi, \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \psi \right\rangle - \nabla_{e_i} \left\langle \varphi, \sum_{i=1}^{n} e_i \cdot \psi \right\rangle \\
&= \langle \varphi, \not{D}\psi \rangle - \nabla_{e_i} \left\langle \varphi, \sum_{i=1}^{n} e_i \cdot \psi \right\rangle
\end{aligned}$$

hence, by integrating over a large enough domain containing the support of $\langle \varphi, \sum_{i=1}^{n} e_i \cdot \psi \rangle$, by the divergence theorem we get that

$$\left\langle\left\langle \not{D}\varphi, \psi \right\rangle\right\rangle = \left\langle\left\langle \varphi, \not{D}\psi \right\rangle\right\rangle$$

hence concluding the proof. □

The following theorem is a very important result concerning the Dirac operator.

Theorem 2.18 (Lichnerowicz formula)**.** *Let (M, g) be a Riemannian spin manifold, and let $S(M)$ be a spinor bundle as above. Then $\forall \psi \in \Gamma(S(M))$*

$$\not{D}^2 \psi = \nabla^* \nabla \psi + \frac{1}{4} R \psi$$

where ∇^* is the formal adjoint of ∇ on $S(M)$.

A proof of this statement in a style close to the one of this essay can be found in reference [Lee19]; a proof with more context can be found in reference [LM90].

3 Asymptotically euclidean analysis

In this chapter we introduce the basics of analysis on asymptotically euclidean manifolds, as it will be necessary for the proof of the positive mass theorem. The references, as for most of the rest of this essay, are [LP87] and [Lee19].

Our ultimate goal is to solve the equation $\not{D}\psi = \eta$ on spinor bundles, but more in general asymptotically euclidean analysis deals with solving differential equations such as the Poisson equation $\Delta u = f$ on asymptotically euclidean manifolds. To accomplish this, we want to find subsets of Lebesgue and Sobolev spaces on such manifolds that behave nicely on the asymptotic ends. For example, we want functions to decay to zero with an enough fast rate on such ends. This is realized precisely through the definition of weighted Sobolev and Lebesgue spaces. In this chapter and in the end of the essay, as defined in the appendix, the product $\langle\!\langle , \rangle\!\rangle$ will denote the product $\langle , \rangle$ integrated over the whole manifold

$$\langle\!\langle \eta, \varphi \rangle\!\rangle \stackrel{\text{def}}{:=} \int_M \langle \eta, \varphi \rangle dV_g,$$

which is the inner product of the Hilbert space $L^2(E)$.

For the purpose of such definition let N be our asymptotically flat manifold and let $r \in C^\infty(N)$ be a smooth function such that $r = |x|$ on each asymptotic end for some chosen asymptotic coordinates. For simplicity in calculations it is handy to deform the function r slightly on a suitable compact set in such a way that it globally takes value in $[1, +\infty)$.

Definition 3.1. Let $p \in [1, +\infty)$ and let $s \in \mathbb{R}$. Let E be a Riemannian vector bundle over N. Then we define the weighted Lebesgue space $L^p_s(E)$ tu be the space of all sections $\varphi \in L^p_{\text{loc}}(E)$ for which the norm

$$\|\varphi\|_{L^p_s(E)} = \left(\int_N |\varphi|^p r^{-sp-n} dV_g \right)^{1/p}$$

is finite. In case $p = \infty$ we define $\|\varphi\|_{L^\infty_s(E)} = \|r^s \varphi\|_{L^\infty(E)}$.

We can give a similar definition also in the case of Sobolev spaces.

Definition 3.2. Let $k \in \mathbb{N}^+$ and let E be as above. We define the weighted Sobolev space $W^{w,k}_s(E)$ to be the space of sections $\varphi \in W^{k,p}_{\text{loc}}(E)$ such that the norm

$$\|\varphi\|_{W^{k,p}_s(E)} = \sum_{i=0}^{k} \left\| \nabla^i \varphi \right\|_{L^p_{s-i}(E)}$$

is finite.

The spaces $L^p(E)$ and $W^{k,p}(E)$ are defined in section A.3. Note that the above definitions perfectly work also in the case of the trivial 1-dimensional bundle, namely in the case of functions. A case of particular interest in this essay is that of $W^{1,2}_{-q}(E)$, whose norm is

$$\|\varphi\|_{W^{1,2}_{-q}} = \|\varphi\|_{L^2_{-q}} + \|\nabla\varphi\|_{L^2_{-q-1}}.$$

Clearly we hope that the above definitions do not depend on the choice of r. This and even more is true in the cases that interest us as the following proposition shows.

Proposition 3.1. *A different choice asymptotically flat metric with the same order or a different choice of the function r will produce equivalent norms in the spaces $L^p_s(E)$, $W^{1,p}_s(E)$ and $W^{2,p}_s(E)$ for all s and p. As a consequence, membership of sections in these spaces does not depend on g or on the choice of r.*

The spaces $L^2_s(E)$ and $W^{k,2}_s(E)$ are special because they are Hilbert spaces with the appropriate products. $L^2_s(E)$ has product $\langle\!\langle , \rangle\!\rangle_{L^2_s}$ defined by

$$\forall \varphi, \psi \in L^2_s \quad \langle\!\langle \varphi, \psi \rangle\!\rangle_{L^2_s} \stackrel{\text{def}}{:=} \int_M \langle \varphi, \psi \rangle r^{-2s-n} dV_g = \langle\!\langle r^{-2s-n}\varphi, \psi \rangle\!\rangle,$$

whereas $W^{k,2}_s$ has product $\langle\!\langle , \rangle\!\rangle_{W^{k,2}_s}$ defined by

$$\forall \varphi, \psi \in W^{k,2}_s \quad \langle\!\langle \varphi, \psi \rangle\!\rangle_{W^{k,2}_s} \stackrel{\text{def}}{:=} \sum_{i=0}^{k} \langle\!\langle \nabla^i \varphi, \nabla^i \psi \rangle\!\rangle_{L^2_{s-k}}.$$

Now let $q = \frac{n-2}{2}$. By the above discussion, the space $W^{1,2}_{-q}$ is a well-defined Hilbert space, independent of the metric and of the choice of r with inner product

$$\forall \varphi, \psi \in W^{1,2}_{-q} \quad \langle\!\langle \varphi, \psi \rangle\!\rangle_{W^{1,2}_{-q}} \stackrel{\text{def}}{:=} \langle\!\langle \varphi, \psi \rangle\!\rangle_{L^2_{-q}} + \langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle = \left\langle\!\!\left\langle \frac{\varphi}{r}, \frac{\psi}{r} \right\rangle\!\!\right\rangle + \langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle .$$

Indeed, note that for such q, by definition 3.1, $L^2_{-q-1} = L^2$.

To prove our next result, we will need an analogous of Poincaré inequality for Sobolev spaces in the case of weighted Sobolev spaces.

Theorem 3.2 (Weighted Poincaré inequality)**.** *Let (N, g) be a complete asymptotically flat manifold and let E be a Riemannian vector bundle. Moreover, let $p \in [1, +\infty)$ and let $s \in (-\infty, 0)$. Then there is a constant $C \in \mathbb{R}^+$ such that for all $\varphi \in W^{k,p}_s(E)$*

$$\|\varphi\|_{L^p_s} \leq C \|\nabla\varphi\|_{L^p_{s-1}}$$

The proof can be found in reference [Lee19].

3.1 Asymptotically euclidean analysis on spin manifolds

We can finally prove our goal theorem, which exhaustively addresses the problem of solving the equation $D\!\!\!/\psi = \eta$, assuming enough regularity for ψ and η. Note that by lemma 2.17, the Dirac operator is formally self-adjoint over whatever space has the property that the intersection with the subset of smooth sections is dense, such as $L^2(S(M))$.

Theorem 3.3. *Let (N, g) be a complete asymptotically euclidean spin manifold of dimension ≥ 3 with nonnegative scalar curvature, and suppose the order is strictly greater than $q = \frac{n-2}{2}$. Then the Dirac operator $D\!\!\!/ : W^{1,2}_{-q}(S(N)) \longrightarrow L^2_{-q-1}(S(N))$ is an isomorphism of topological vector spaces.*

Proof. Being a differential operator, it is clear that it is linear. To see it is bounded, let $\varphi \in W^{1,2}_{-q}(S(N))$

$$\begin{aligned} \|D\!\!\!/\varphi\|_{L^2_{-q-1}} &= \left\| \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi \right\|_{L^2_{-q-1}} \leq \sum_{i=1}^{n} \|e_i \cdot \nabla_{e_i}\psi\|_{L^2_{-q-1}} \\ &= \sum_{i=1}^{n} \|\nabla_{e_i}\psi\|_{L^2_{-q-1}} \leq \|\nabla\psi\|_{L^2_{-q-1}} \leq \|\varphi\|_{W^{1,2}_{-q}} \end{aligned}$$

where we used that e_i acts orthogonally. $\sum_{i=1}^{n} \|\nabla_{e_i}\psi\|_{L^2_{-q-1}} \leq \|\nabla\psi\|_{L^2_{-q-1}}$ follows from lemma A.6.

Next we want to prove that $\not{D}$ is injective. To do so we prove it is bounded from below. By proposition 4.2 and corollary 4.4, by choosing $\psi_0 = 0$, we have that if $\varphi \in W^{1,2}_{-q}(S(N))$, then

$$\int_\Omega \left(|\nabla\varphi|^2 - |\not{D}\varphi|^2 + \frac{1}{4}R|\varphi|^2\right) dV_g = 0.$$

Hence, since by hypothesis that R is nonnegative, we have that $\|\nabla\varphi\|_{L^2} \leq \|\not{D}\varphi\|_{L^2}$, which as we said above is the same as $\|\nabla\varphi\|_{L^2_{-q-1}} \leq \|\not{D}\varphi\|_{L^2_{-q-1}}$. By the weighted Poincaré inequality 3.2 there exists a constant $C \in \mathbb{R}^+$ such that

$$\|\varphi\|_{L^2_{-q}} \leq C\|\nabla\varphi\|_{L^2_{-q-1}}.$$

Hence

$$\begin{aligned} \|\varphi\|_{W^{1,2}_{-q}} &= \|\varphi\|_{L^2_{-q}} + \|\nabla\varphi\|_{L^2_{-q-1}} \\ &\leq (C+1)\|\not{D}\varphi\|_{L^2}. \end{aligned} \tag{3.1}$$

Hence it is injective, as if $\not{D}\varphi = 0$, by the above inequality, $\|\varphi\|_{W^{1,2}_{-q}} = 0$ and hence $\varphi = 0$.

It remains to prove surjectivity. Suppose $\eta \in L^2(S(N))$. We want to find $\psi \in W^{1,2}_{-q}(S(N))$ such that $\not{D}\psi = \eta$. Start by supposing η to be compactly supported. Define the inner product $\langle\!\langle \omega, \varphi \rangle\!\rangle_{\not{D}} \overset{\text{def}}{:=} \left\langle\!\left\langle \not{D}\omega, \not{D}\varphi \right\rangle\!\right\rangle$. The estimate in equation 3.1 together with the boundness of $\not{D}$ show that $\langle\!\langle , \rangle\!\rangle_{W^{1,2}_{-q}}$ and $\langle\!\langle , \rangle\!\rangle_{\not{D}}$ are equivalent (meaning their induced norms are). Now, the map

$$\begin{aligned} W^{1,2}_{-q}(S(N)) &\to \mathbb{R} \\ \varphi &\mapsto \langle\!\langle \eta, \varphi \rangle\!\rangle \end{aligned}$$

is a bounded linear functional. It is well defined as $W^{1,2}_{-q}(S(N)) \subset L^2(S(N))$. To see it is bounded we need to use the fact that η is compactly supported. Indeed, let $K \overset{\text{def}}{:=} \operatorname{Supp}\eta$ consider that for some $C \in \mathbb{R}^+$

$$|\langle\!\langle \eta, \varphi \rangle\!\rangle| \leq \|\eta\|_{L^2(K)} \|\varphi\|_{L^2(K)} \leq C \|\eta\|_{L^2} \|\varphi\|_{L^2_{-q-1}(K)} \leq C \|\eta\|_{L^2} \|\varphi\|_{W^{1,2}_{-q}(K)} \leq \|\eta\|_{L^2} \|\varphi\|_{W^{1,2}_{-q}}.$$

The only non-trivial bound is $\|\varphi\|_{L^2(K)} \leq C \|\varphi\|_{L^2_{-q-1}(K)}$, which is valid because on a compact set all weighted Lebesgue spaces are the same, meaning that they contain the same functions and the norms are equivalent. Indeed, as the function r is smooth, it has a maximum and a minimum on K and the minimum is ≥ 1 because of where we required the function to take values. This proves that all norms are equivalent to the L^2 norm and hence they are all equivalent to each others by transitivity. The fact that all spaces contain the same functions in this case follows by how the they are defined. By the Riesz representation theorem applied to the space $\left(W^{1,2}_{-q}(S(N)), \langle\!\langle , \rangle\!\rangle_{\not{D}}\right)$ (as the product $\langle\!\langle , \rangle\!\rangle_{\not{D}}$ is equivalent to the native, the functional is bounded with respect to this product as well), there must exist $\omega \in W^{1,2}_{-q}(S(N))$ such that

$$\forall\varphi \in W^{1,2}_{-q}(S(N)) \quad \left\langle\!\left\langle \not{D}\omega, \not{D}\varphi \right\rangle\!\right\rangle = \langle\!\langle \eta, \varphi \rangle\!\rangle.$$

We want to prove that $\psi = \not{D}\omega$.

By now, all what we can say about the regularity of ψ is that $\psi \in L^2(S(N))$, but we need $\psi \in W^{1,2}_{-q}(S(N))$. To prove so let ψ_k be a sequence of spinors converging to ψ in L^2. By construction of ψ, we have that for any $\varphi \in W^{1,2}_{-q}(S(N))$

$$\lim_{k\to\infty} \left\langle\!\left\langle \not{D}\psi_k, \varphi \right\rangle\!\right\rangle = \lim_{k\to\infty} \left\langle\!\left\langle \psi_k, \not{D}\varphi \right\rangle\!\right\rangle = \left\langle\!\left\langle \psi, \not{D}\varphi \right\rangle\!\right\rangle = \langle\!\langle \eta, \varphi \rangle\!\rangle$$

and hence $\not{D}\psi_k$ converges to η in the weak topology of L^2. As a consequence, $\left\|\not{D}\psi_k\right\|_{L^2}$ is bounded independently of k, and by equation 3.1 also $\|\psi_k\|_{W^{1,2}_{-q}}$ is bounded. Hence, ψ_k must converge weakly to ψ in $W^{1,2}_{-q}(S(M))$ as well.

Now, to see that ψ satisfies our equation, we note that for all compactly supported spinors $\varphi \in W^{1,2}_{-q}(S(N))$ it holds

$$\left\langle\!\left\langle \not{D}\psi, \varphi \right\rangle\!\right\rangle = \left\langle\!\left\langle \psi, \not{D}\varphi \right\rangle\!\right\rangle = \langle\!\langle \eta, \varphi \rangle\!\rangle$$

and hence $\not{D}\psi = \eta$.

Finally we consider the case of a generic $\eta \in L^2(S(N))$ not necessarily compactly supported. In such case let η_k be a sequence of compactly supported converging to η in L^2, and let ψ_k be the solutions to $\not{D}\psi_k = \eta_k$. By what we saw above $\psi_k \in W^{1,2}_{-q}(S(N))$. Because of equation 3.1 ψ_k are bounded in $W^{1,2}_{-q}(S(N))$ and hence they converge to $\psi \in W^{1,2}_{-q}(S(N))$. It follows that $\not{D}\psi = \eta$. □

4 Proof of the positive mass theorem

In this chapter we finally go through the proof of the positive mass theorem. As anticipated, this proof does not cover all cases as the original theorem, but only the case of spin manifolds. We will then be able to extend it to some other manifolds, but all the same not to all cases of the theorem proved by Schoen and Yau. Here is a precise statement of the theorem we are going to prove.

Theorem 4.1 (Positive mass theorem for spin manifolds)**.** *Let (N, g) be a complete asymptotically euclidean spin n-manifold with nonnegative scalar curvature and $n \geq 3$. Suppose further that N admits ADM mass. Then the ADM mass of each end is nonnegative. Moreover, if the mass of any end is zero, then (N, g) is globally isomorphic to the euclidean space.*

We start with the following proposition, which will allow us to take advantage of the Lichnerowicz formula on asymptotically euclidean manifolds.

Proposition 4.2. *Let $\Omega \subseteq N$ be a bounded subset with smooth boundary, where N is a complete spin manifold. Then, $\forall \psi \in \Gamma(S(N))$*

$$\int_\Omega \left(|\nabla\psi|^2 - |\not{D}\psi|^2 + \frac{1}{4}R|\psi|^2 \right) dV_g = \int_{\partial\Omega} \sum_{i=1}^n \langle \psi, L_i\psi \rangle \, \nu^i dV_g$$

where ν is the normal unit vector to $\partial\Omega$ and $L_i = \sum_{j \neq i} e_i e_j \cdot \nabla_j$.

Proof. By using Einstein summation convention even on indices in the same position (down-down) and by writing $\nabla_i = \nabla_{e_i}$ we perform the calculation term by term:

$$\begin{aligned}
-|\not{D}\psi|^2 &= -\langle e_i \cdot \nabla_i\psi, e_j \cdot \nabla_j\psi \rangle \\
&= -\langle \nabla_i(e_i \cdot \psi), e_j \cdot \nabla_j\psi \rangle + \langle \nabla_i(e_i) \cdot \psi, e_j \cdot \nabla_j\psi \rangle \\
&= \nabla_i \langle -e_i \cdot \psi, e_j \cdot \nabla_j\psi \rangle + \langle e_i \cdot \psi, \nabla_i (e_j \cdot \nabla_j\psi) \rangle + \langle \nabla_i(e_i) \cdot \psi, e_j \cdot \nabla_j\psi \rangle \\
&= \nabla_i \langle \psi, e_i e_j \cdot \nabla_j\psi \rangle - \langle \psi, e_i \cdot \nabla_i (e_j \cdot \nabla_j)\psi \rangle + \langle \nabla_i(e_i) \cdot \psi, e_j \cdot \nabla_j\psi \rangle \\
&= \nabla_i \langle \psi, e_i e_j \cdot \nabla_j\psi \rangle - \left\langle \psi, \not{D}^2\psi \right\rangle + \langle \nabla_i(e_i) \cdot \psi, e_j \cdot \nabla_j\psi \rangle
\end{aligned}$$

where we made extensive use of proposition 2.15. We can forget about the last term by taking an orthonormal frame parallel at a point and trying to prove that the other terms in the equation do not depend on the choice of the local orthonormal frame. The only term which could depend on the choice of coordinates is $\nabla_i \langle \psi, e_i e_j \cdot \nabla_j\psi \rangle$, but indeed we have that $\nabla_i \langle \psi, e_i e_j \cdot \nabla_j\psi \rangle = \operatorname{div} V$ where V is defined by

$$V = \sum_{i=1}^n \langle \psi, e_i \cdot \not{D}\psi \rangle e_i.$$

So it suffices to prove that V is independent on the frame. Indeed let $\{\tilde{e}_j\}$ be another frame and let $\tilde{e}_j = A_i^{\ j} e_j$. Call B the inverse of A. Then

$$\begin{aligned}
\sum_{i=1}^n \langle \psi, e_i \cdot \not{D}\psi \rangle e_i &= \sum_{i=1}^n \langle \psi, \delta_i^{\ \ell} e_\ell \cdot \not{D}\psi \rangle \delta_i^{\ j} e_j = \sum_{i=1}^n \langle \psi, B_i^{\ m} A_m^{\ \ell} e_\ell \cdot \not{D}\psi \rangle B_i^{\ k} A_k^{\ j} e_j \\
&= \sum_{i=1}^n B_i^{\ k} B_i^{\ m} \langle \psi, \tilde{e}_m \cdot \not{D}\psi \rangle \tilde{e}_j = \delta^{mj} \langle \psi, \tilde{e}_m \cdot \not{D}\psi \rangle \tilde{e}_j = \sum_{i=1}^n \langle \psi, \tilde{e}_i \cdot \not{D}\psi \rangle \tilde{e}_i.
\end{aligned}$$

where we used that B is an element of $O(n)$.

In the meantime

$$\begin{aligned}|\nabla\psi|^2 = \langle\nabla_i\psi,\nabla_i\psi\rangle &= \nabla_i\langle\psi,\nabla_i\psi\rangle + \langle\psi,\nabla^*\nabla\psi\rangle \\ &= \nabla_i\langle\psi,\delta_{ij}\nabla_j\psi\rangle + \langle\psi,\nabla^*\nabla\psi\rangle\end{aligned}$$

where we used proposition A.7. Hence, by making use of the Lichnerowicz formula (theorem 2.18),

$$\begin{aligned}|\nabla\psi|^2 - |\not{D}\psi|^2 + \frac{1}{4}R|\psi|^2 &= \nabla_i\langle\psi,\delta_{ij}\nabla_j\psi\rangle + \langle\psi,\nabla^*\nabla\psi\rangle + \nabla_i\langle\psi,e_ie_j\cdot\nabla_j\psi\rangle \\ &\quad - \left\langle\psi,\not{D}^2\psi\right\rangle + \frac{1}{4}R\langle\psi,\psi\rangle \\ &= \nabla_i\langle\psi,\delta_{ij}\nabla_j\psi\rangle + \nabla_i\langle\psi,e_ie_j\cdot\nabla_j\psi\rangle = \nabla_i\langle\psi,L_i\psi\rangle.\end{aligned}$$

Indeed note that

$$L_i = \sum_{j\neq i} e_ie_j\cdot\nabla_j = \nabla_i + \sum_{j=1}^{n} e_ie_j\cdot\nabla_j = \sum_{j=1}^{n}(\delta_{ij}+e_ie_j)\nabla_j.$$

Now the result follows by the divergence theorem. Note that we can apply the divergence theorem because the manifold is complete and hence Ω has compact closure and any form defined on Ω is compactly supported. □

The idea behind Witten's proof is to find a harmonic spinor which is asymptotically constant at infinity. In such case the term on the right in the above proposition turns out to be proportional to the mass, and the term on the left is positive, being the only negative term zero. The next step in the proof is hence to show that the term on the right is proportional to the mass under the mentioned hypotheses.

Lemma 4.3. *Let (N,g) be an asymptotically euclidean spin manifold of dimension greater than or equal to 3, that also admits ADM mass. Let $e_1,...,e_n$ be the orthogonal frame defined by orthonormalising via Gram-Schmidt the coordinate frame on some asymptotic end N_k and let ψ_0 be a constant spinor with respect to such frame. Then*

$$\lim_{R\to\infty}\int_{S^{n-1}_{R,k}}\sum_{i=1}^{n}\langle\psi_0,L_i\psi_0\rangle\,\nu^i dV_g = \frac{1}{2}(n-1)\,\mathrm{Vol}(S^{n-1})\,|\psi_0|^2\,m_{\mathrm{ADM}}(N_k,g)$$

where $S^{n-1}_{R,k}$ is a coordinate sphere in the end N_k as in chapter 1.

Before starting the proof, we will introduce a new notation which will come in handy later in the proof. We say that $f\in C^k(\mathbb{R})$ is $o_k(g(x))$ if there exists $R\in\mathbb{R}^+$ such that if $|x|>R$

$$\sum_{i=0}^{k}\left|x^i\frac{d^if}{dx^i}\right|\leq cg(x)$$

for some $c\in\mathbb{R}^+$.

Proof. Let $q=\frac{n-2}{2}$. We start by rewriting the integrand on the left

$$\begin{aligned}\sum_{i=1}^{n}\langle\psi_0,L_i\psi_0\rangle\,\nu^i &= \sum_{i\neq j}\langle\psi_0,e_ie_j\nabla_j\psi_0\rangle\,\nu^i \\ &= \sum_{i\neq j}\left\langle\psi_0,e_ie_j\frac{1}{4}\sum_{k\neq\ell}\omega^k_\ell(e_j)\,e_ke_\ell\cdot\psi_0\right\rangle\nu^i \qquad (4.1)\\ &= \frac{1}{4}\sum_{\substack{i\neq j\\ k\neq\ell}}\omega^k_\ell(e_j)\,\langle\psi_0,e_ie_je_ke_\ell\cdot\psi_0\rangle\,\nu^i\end{aligned}$$

where ω^i_j are again the connection 1-forms determined by the frame. To work them out we find an asymptotic expression for the elements of the frame e_i. The hypothesis that the manifold admits ADM mass, tells us in particular that the decay rate is greater than q. Hence $h_{ij} = g_{ij} - \delta_{ij} = o_2(|x|^{-q})$ (it might be true with a higher exponent but we just need q). A direct calculation shows that

$$e_i = \partial_i - \frac{1}{2} h_{ij}\partial_j + o_1\left(|x|^{-q}\right)$$

where ∂_j denote the vectors induced by the coordinates. Thus

$$\begin{aligned}
\omega^i_j\left(e_k\right) &= \langle \nabla_{e_k} e_j, e_i\rangle \\
&= \left\langle \nabla_{\partial_k}\left(\partial_j + \frac{1}{2}\sum_{\ell=1}^{n} h_{j\ell}\partial_\ell\right), \partial_i\right\rangle + o\left(|x|^{-2q-1}\right) \\
&= \Gamma^i_{jk} - \frac{1}{2} h_{ij,k} + o\left(|x|^{-2q-1}\right) \\
&= \frac{1}{2}\left(g_{ik,j} - g_{jk,i}\right) + o\left(|x|^{-2q-1}\right)
\end{aligned}$$

This expression shows that

$$\omega^i_j\left(e_k\right) + \omega^j_k\left(e_i\right) + \omega^k_i\left(e_j\right) = o\left(|x|^{-2q-1}\right)$$

Note now that if i, j and k are distinct $e_i e_j e_k$ is invariant under cyclic permutations. Hence

$$\begin{aligned}
\sum_{j,k,\ell \text{ distinct}} \omega^k_\ell(e_j) e_j e_k e_\ell &= \frac{1}{3}\sum_{j,k,\ell \text{ distinct}} \omega^k_\ell(e_j) e_j e_k e_\ell + \omega^\ell_j(e_k) e_k e_\ell e_j + \omega^j_k(e_\ell) e_\ell e_j e_k \\
&= \frac{1}{3}\sum_{j,k,\ell \text{ distinct}} \left(\omega^k_\ell(e_j) + \omega^\ell_j(e_k) + \omega^j_k(e_\ell)\right) e_j e_k e_\ell \\
&= o\left(|x|^{-2q-1}\right)
\end{aligned}$$

and hence, in equation 4.1 we only need to worry about terms where either $j = k$ or $j = \ell$. We can get rid of even more terms. Indeed, the terms with $j = k$ and $i \neq \ell$ vanish, because in that case we are left with $\langle \psi_0, e_i e_\ell \cdot \psi_0\rangle$ that vanishes because $e_i e_\ell$ is skew-symmetric (indeed the product of skew-symmetric operators that anticommute is skew-symmetric). Similarly, we can also forget about the terms for which $j = \ell$ and $i \neq k$. Thus, we are left only with the terms for which either $j = k$ and $i = \ell$, or $j = \ell$ and $i = k$. These are few enough terms that we can write them all down.

$$\begin{aligned}
\sum_{i=1}^{n} \langle \psi_0, L_i \psi_0\rangle \nu^i =& \frac{1}{4}\sum_{i\neq j}\Big(\omega^i_j\left(e_j\right)\langle \psi_0, e_i e_j e_i e_j \cdot \psi_0\rangle + \omega^j_i\left(e_j\right)\langle \psi_0, e_i e_j e_j e_i \cdot \psi_0\rangle\Big)\nu^i \\
&+ o\left(|x|^{-2q-1}\right) \\
=& \frac{1}{8}\sum_{i\neq j}\left(-\left(g_{ij,i} - g_{jj,i}\right) + \left(g_{jj,i} - g_{ij,j}\right)\right)|\psi_0|^2 \nu^i + o\left(|x|^{-2q-1}\right) \\
=& \frac{|\psi_0|^2}{4}\sum_{i,j}\left(g_{jj,i} - g_{ij,j}\right)\hat{n}^i + o\left(|x|^{-2q-1}\right) \\
=& \frac{|\psi_0|^2}{4}\left(g^j{}_{j,i} - g_{ij,}{}^j\right)\hat{n}^i + o\left(|x|^{-2q-1}\right) \\
=& -\frac{|\psi_0|^2}{4}\bar{\xi}_g(\hat{n}) + o\left(|x|^{-2q-1}\right)
\end{aligned}$$

where with $\hat{n}$ we mean the normal vector calculated with the euclidean metric instead of the pullback of g and $\bar{\xi}_g$ is the 1-form defined in lemma 1.2. When integrating, the term $o\left(|x|^{-2q-1}\right)$

decays fast enough that its integral vanishes in the limit. Indeed, $2q + 1 = n - 1$ and as $\mathrm{Vol}\left(S_R^{n-1}\right) \sim r^{n-1}$ whatever is $o\left(|x|^{-(n-1)}\right)$ vanishes in the limit. By lemma 1.2 we get the thesis. □

The next step in the proof of the positive mass theorem is to relax the hypothesis of ψ_0 being constant to it being "asymptotically constant".

Corollary 4.4. *Under the same hypothesis of the above proposition, if $\psi \in \Gamma(S(M))$ is such that $\psi - \psi_0 \in W^{1,2}_{-q}(S(M))$ where $q = \frac{n-2}{2}$ as above, then*

$$\lim_{R\to\infty} \int_{S^{n-1}_{R,k}} \sum_{i=1}^{n} \langle \psi, L_i\psi\rangle \nu^i dV_g = \frac{1}{2}(n-1)\,\mathrm{Vol}\left(S^{n-1}\right) |\psi_0|^2\, m_{\mathrm{ADM}}\left(M_k, g\right).$$

Proof. Let $\chi = \psi - \psi_0$. We write the product $\langle \psi, L_i\psi\rangle$ as

$$\langle \psi, L_i\psi\rangle = \langle \psi_0, L_i\psi_0\rangle + \langle \psi_0, L_i\chi\rangle + \langle \chi, L_i\psi_0\rangle + \langle \chi, L_i\chi\rangle$$

Summing and integrating the first term on the right produces the right side of the equation by the previous proposition. Hence, we are done if we prove that the other terms do not contribute in the limit $R \to \infty$. We start from the last term. As χ has decay rate more than q and $\nabla\chi$ has decay rate more than $q+1$, the total decay rate is more than $2q+1 = n-1$ and hence this term does not contribute. Moreover, in the proof of the previous proposition we saw that $\nabla\psi_0$ depends on $\bar{\xi}_g$ and hence has decay rate more than $q+1$, being the manifold asymptotically flat of order greater than q. Hence, the decay rate of the term $\langle \chi, L_i\psi_0\rangle$ is greater than $2q+1$ and again this term does not contribute in the limit. To see that the term $\langle \psi_0, L_i\chi\rangle$ does not contribute either is a bit harder as ψ_0 does not decay at big distances. To prove so we can try to "integrate it by parts". Define α to be the $n-2$ form

$$\alpha = \sum_{i\neq j} \langle \psi_0, e_i e_j \xi\rangle\, e_i \lrcorner\, e_j \lrcorner\, dV_g$$

Then

$$\begin{aligned} d\alpha &= 2\sum_{i\neq j} \left(-\nabla_j \langle \psi_0, e_i e_j \xi\rangle\right) e_i \lrcorner\, dV_g \\ &= 2\sum_{i\neq j} \left(\langle e_i e_j \nabla_j \psi_0, \xi\rangle - \langle \psi_0, e_i e_j \nabla_j \xi\rangle\right) e_i \lrcorner\, dV_g \\ &= 2\sum_{i} \left(\langle L_i\psi_0, \xi\rangle - \langle \psi_0, L_i\xi\rangle\right) e_i \lrcorner\, dV_g \\ &= 2\sum_{i} \left(\langle L_i\psi_0, \xi\rangle - \langle \psi_0, L_i\xi\rangle\right) \nu^i dV_{S_R^{n-1}} \end{aligned}$$

where we used skew-symmetry of $e_i e_j$ and where in the last line we used lemma 1.3. Since $d\alpha$ is exact, we have that $\int_{S_R^{n-1}} d\alpha = 0$, and hence the integral of $\langle L_i\psi_0, \xi\rangle$ is equal to the integral of $\langle \psi_0, L_i\xi\rangle$, which we saw to be zero in the limit of big R. □

Finally, we now have all the necessary machinery to approach the proof of the positive mass theorem for spin manifolds.

Proof of theorem 4.1. Let N_k be an asymptotic end of the manifold and let $e_1, ..., e_n$ be the orthonormal frame on such end obtained by orthonormalizing the coordinate frame. Let $\psi_0 \in \Gamma(S(N))$ be constant spinor field with respect to the frame $e_1, ..., e_n$ and $|\psi_0| = 1$ on N_k. Moreover, suppose it vanishes on other ends.

Let $\eta = -\not{D}\psi_0$. Let us see that $\eta \in L^2(S(M))$. By proposition 4.2 and lemma 4.3 we have that

$$\|\nabla\psi_0\|_{L^2} + \frac{1}{4}\int_N R|\psi_0|^2 dV_g = \left\|\not{D}\psi_0\right\|_{L^2} + \frac{1}{2}(n-1)\,\mathrm{Vol}\left(S^{n-1}\right)|\psi_0|^2\, m_{\mathrm{ADM}}\left(N_k, g\right).$$

We do not know yet the mass is positive but we know it is finite, and this is enough to make an estimate. Moreover as $|\psi_0| = 1$ we have that

$$\left\|\not{D}\psi_0\right\|_{L^2} \leq \|\nabla\psi_0\|_{L^2} + \frac{1}{4}\int_N R dV_g + \frac{1}{2}(n-1)\operatorname{Vol}\left(S^{n-1}\right)\left|m_{\mathrm{ADM}}\left(N_k, g\right)\right|.$$

Now, as we said the mass is finite and $\int_N R dV_g$ is finite because of the hypothesis that N admits ADM mass. Hence, $\not{D}\psi_0$ is in L^2 if $\nabla\psi_0$ is, but this is indeed the case as can be seen by its explicit expression 2.3.

As $\eta \in L^2(S(M))$, by proposition 3.3 there exists $\varphi \in W^{1,2}_{-q}(S(M))$ such that $\not{D}\varphi = \eta$ ($q = \frac{n-2}{2}$ as always). Let $\psi \overset{\text{def}}{:=} \psi_0 + \varphi$. By proposition 4.2 and lemma 4.4 we have

$$\int_M \left(|\nabla\psi|^2 - |\not{D}\psi|^2 + \frac{1}{4}R\psi^2\right) dV_g = \frac{1}{2}(n-1)\operatorname{Vol}\left(S^{n-1}\right) m_{\mathrm{ADM}}\left(N_k, g\right).$$

As

$$\not{D}\psi = \not{D}\psi_0 + \not{D}\varphi = \eta - \eta = 0$$

it follows that

$$m_{\mathrm{ADM}}\left(N_k, g\right) = \frac{2}{(n-1)\operatorname{Vol}\left(S^{n-1}\right)}\int_M \left(|\nabla\psi|^2 + \frac{1}{4}R\psi^2\right) dV_g \tag{4.2}$$

which is nonnegative if $R \geq 0$.

It remains to prove the positive mass rigidity, that is to say that if the mass of one end is 0 than the manifold is euclidean. Suppose then that $m_{\mathrm{ADM}}(N_k, g) = 0$. Equation (4.2) implies that $\nabla\psi = 0$ everywhere, meaning ψ is parallel everywhere. Now, repeat the construction done above for ψ for each of the constant spinors $e_i \cdot \psi_0$, obtaining n spinors $\psi_1, ..., \psi_n$. For each ψ_i we have $\nabla\psi_i = 0$ everywhere. By the representation theorem we can construct the vector fields V_i with the property that

$$\langle V_i, w\rangle = \langle w \cdot \psi, \psi_i\rangle$$

for any $w \in \mathfrak{X}(N)$, and where ψ is the original parallel spinor constructed above.

Note that for all vector fields $X \in \mathfrak{X}(N)$ we have that

$$\begin{aligned}
\langle \nabla_X V_i, w\rangle &= \nabla_X \langle V_i, w\rangle - \langle V_i, \nabla_X w\rangle \\
&= \nabla_X \langle w \cdot \psi, \psi_i\rangle - \langle \nabla_X(w) \cdot \psi, \psi_i\rangle \\
&= \langle \nabla_X(w \cdot \psi), \psi_i\rangle + \langle w \cdot \psi, \nabla_X \psi_i\rangle - \langle \nabla_X(w) \cdot \psi, \psi_i\rangle \\
&= \langle \nabla_X(w) \cdot \psi, \psi_i\rangle + \langle w \cdot \nabla_X(\psi), \psi_i\rangle - \langle \nabla_X(w) \cdot \psi, \psi_i\rangle = 0
\end{aligned}$$

and thus $\nabla V_i = 0$ everywhere.

Now we prove that $\{V_i\}$ tends asymptotically to the basis $\{e_i\}$. To see we decompose it on the orthonormal basis $\{e_i\}$

$$\langle V_i, e_j\rangle = \langle e_j \cdot \psi, \psi_i\rangle = \langle e_j \cdot \varphi + e_j \cdot \psi_0, e_i \cdot \psi_0\rangle = \langle e_j \cdot \varphi, e_i \cdot \psi_0\rangle + \langle e_j \cdot \psi_0, e_i \cdot \psi_0\rangle.$$

The first term vanishes at infinity as $\varphi \in W^{1,2}_{-q}(S(M))$. The second depends on j. If $j = i$ then

$$\langle e_i \cdot \psi_0, e_i \cdot \psi_0\rangle = \langle \psi_0, \psi_0\rangle = |\psi_0|^2 = 1$$

If $j \neq i$ then $e_i e_j = -e_j e_i$ and hence

$$\langle e_j \cdot \psi_0, e_i \cdot \psi_0\rangle = -\langle \psi_0, e_j e_i \cdot \psi_0\rangle = \langle \psi_0, e_i e_j \cdot \psi_0\rangle = -\langle e_i\psi_0, e_j \cdot \psi_0\rangle = -\langle e_j\psi_0, e_i \cdot \psi_0\rangle$$

and thus $\langle e_j \cdot \psi_0, e_i \cdot \psi_0\rangle = 0$.

Hence, V_i is a global basis of parallel vector fields. Let us see that this implies that the end is euclidean. As $\forall X$

$$\nabla_X\langle V_i, V_j\rangle = \langle \nabla_X V_i, V_j\rangle + \langle V_i, \nabla_X V_j\rangle = 0$$

$\langle V_i, V_j\rangle$ is constant. Moreover, as it is equal to δ_{ij} in the limit, $\{V_i\}$ are orthogonal everywhere by continuity. Now, as the Levi-Civita connection is torsion-free

$$[V_i, V_j] = \nabla_{V_i} V_j - \nabla_{V_j} V_i = 0$$

and hence they are integrable locally to a chart which is an isometry because of the equation $\langle V_i, V_j\rangle = \delta_{ij}$. This proves that the manifold is locally isomorphic to the euclidean space, which is equivalent to being flat.

To prove it is globally isomorphic to the euclidean space we need to rely on the particular topology of the manifold. As the manifold is flat, by the Killing-Hopf theorem, its universal cover is the euclidean space $\mathbb{R}^n$. Hence, if we prove it is simply connected, we can conclude it is globally isometric to its universal cover. Consider the topological boundary of the image of $\mathbb{R}^n \setminus D_{R_k}$ through the chart of the asymptotic end N_k. By definition of asymptotic end, this is an embedded $n-1$-dimensional sphere $\Sigma \subseteq N$. Suppose N were not simply connected. Then the preimage of Σ is made up of at least two components each of which is homeomorphic to Σ. This is because the restriction of a covering map to a subspace is a covering map and if $n \geq 3$ the $n-1$ sphere is simply connected. Both spheres do not bound a compact subset as one of their sides is homeomorphic to $\mathbb{R}^n \setminus D^n$ and on the other side there is the other sphere with another $\mathbb{R}^n \setminus D^n$. Note that this does not necessarily happen if there is only one sphere. Hence, they determine nontrivial elements of $H_{n-1}(\tilde{N})$. But $\tilde{N} \cong \mathbb{R}^n$ is contractible and hence we have a contradiction. The thesis follows as explained above. □

Remark 4.1. Note that the particular topology of asymptotically flat manifolds is essential to prove the positive mass rigidity: the existence of n orthonormal parallel vector fields is not enough since the flat n-torus is an easy counterexample.

The following corollary allows us to extend the result a bit further.

Corollary 4.5. *Let (N, g) be a non-orientable complete asymptotically euclidean n-manifold with nonnegative scalar curvature and $n \geq 3$. Suppose further that N admits ADM mass and that its orientable double cover is a spin manifold. Then the ADM mass of each end is nonnegative. Moreover, if the mass of any end is zero, then (N, g) is globally isomorphic to the euclidean space.*

Proof. Each asymptotic end is covered by two asymptotic ends in the double cover (isometric to the original end) and it suffices to apply the theorem to the orientable double cover. □

This shows that it is not really orientability which prevents the proof to work in the missing cases, but it is really about the vanishing second Stiefel–Whitney class.

4.1 The completeness hypothesis

We conclude this essay with a final remark on the completeness hypothesis that is not usually discussed in surveys on the positive mass theorem, where often such hypothesis is not even explicitly mentioned (it is instead incorporated into the definition of asymptotically euclidean manifold where the set C in our definition is usually required to be compact; this implies that the manifold is complete).

When we defined asymptotically euclidean manifolds we stressed that we had to allow some parts of the definition to be technical in order to include in it non complete manifolds such as Cauchy surfaces of the classical Schwarzschild spacetime. However, one of the hypothesis of the positive mass theorem is that the manifold N is complete, so, not to waste all the effort made to formulate the definition of asymptotically euclidean manifold, it would be natural to ask ourselves if such an hypothesis is really necessary or if we can do without it. We made extensive use of it mainly when we proved the positive mass rigidity, so it would be natural to expect that we would loose that part of the theorem. However, looking back into the proof we might be able to save flatness. We are therefore brought to the following conjecture.

Conjecture 4.6. *Let (N, g) be a generic asymptotically euclidean spin n-manifold with nonnegative scalar curvature and $n \geq 3$. Suppose further that N has a well-defined ADM mass. Then the ADM mass of each end is nonnegative. Moreover, if the mass of any end is zero, then (N, g) is flat.*

Unfortunately, this conjecture is false, and we will have to accept that the positive mass theorem cannot be used for famous examples such as the classical Schwarzschild spacetime. In any case, for the particular case of the black hole, any Cauchy hypersurface of the Kruskal extension does satisfy all the hypotheses of the positive mass theorem (in particular it is complete), so we are able to save this case of interest if we are willing to accept the existence of a parallel universe and of a white hole.

The problem is that the completeness hypothesis is also necessary to use the divergence theorem in lemma 4.2 and to apply theorem 3.3. The problem is that if we do not assume the manifold to be complete the closure of Ω could not be compact. As Ω later becomes implicitly D_R defined in equation 1.8, if it is not complete we should, for large enough R include other boundary terms in the integration corresponding to the terminating geodesics inside D_R (for the Schwarzschild hypersurface for example this would mean adding a term over a sphere around the singularity), and hence the whole argument would be ruined. This shows that the completeness hypothesis is necessary for the whole proof.

It could still be, however, that the above conjecture is true and that it needs a completely different proof. This is not the case, and a counterexample is provided by a Cauchy surface of the negative mass Schwarzschild spacetime. This is the punctured euclidean space with metric

$$ds^2 = -\left(1 + \frac{2m}{r^{n-2}}\right) dt^2 + \left(1 + \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2$$

(where again problems at the event horizon can be solved by switching coordinates). It can be shown that this satisfies all hypothesis of the conjecture but it has negative mass. Clearly such manifold is not complete, and hence the proper positive mass theorem cannot be applied.

A Some useful global analysis

In this appendix we are going to review the basics of differential operators on smooth manifolds, to introduce Sobolev spaces on them and to prove some simple results used in the essay.

A.1 Differential operators on smooth manifolds

The main reference for this section is [Kah80]. To us, a differential operator in the euclidean space is a linear combination of partial derivatives, where coefficients are taken among smooth functions. The *order* of such an operator is just the highest order of the partial derivatives in the linear combination. Because of non-existence of canonical partial derivatives on a smooth manifold, defining what exactly means a differential operator in such a context requires some caution.

Definition A.1. Given an n-dimensional smooth manifold M and two smooth, vector bundles over M, say $E_1 \twoheadrightarrow M$ and $E_2 \twoheadrightarrow M$ let $\Gamma(E_i)$ be the vector space of sections of the bundle. A linear differential operator is a linear map

$$P : \Gamma(E_1) \to \Gamma(E_2)$$

such that

$$\forall s \in \Gamma(E_1) \quad \mathrm{Supp}(P(s)) \subseteq \mathrm{Supp}(s).$$

As the definition is stated more generally for vector bundles, we can consider differential operators on vector fields, differential forms, tensor fields, etc. For example, the exterior derivative, the Lie derivative, the covariant derivative can all be interpreted as differential operators.

The only problem that the given definition may have is that it is not clear what derivations have to do with such general linear maps. We will come to this in a moment. By now, however, we can observe that what we intuitively want to be a differential operator satisfies this definition, as we certainly want it to be a linear map, and moreover if a section σ of the vector field vanishes in a neighbourhood of a point we want its "derivative" to vanish as well, which, switching to complements, is equivalent to require $\mathrm{Supp}(P(\sigma)) \subseteq \mathrm{Supp}(\sigma)$. Moreover, as this proposition shows, differential operators make up a vector space (or more precisely a $C^\infty(M)$-module), which is also something we would intuitively require

Proposition A.1. *Let P and Q be two differential operators on the same vector bundle over a manifold M. Then $\forall g, h \in C^\infty(M)$ $gP + hQ$ is a differential operator. In particular, multiplications with functions ($f \mapsto g \cdot f$) are differential operators. Moreover, the composition of differential operators is a differential operator.*

The proof of such proposition is routine.

Once all these properties have convinced us that this abstract definition might work, we need to show the converse, that is to say that an operator which satisfies our definition is "intuitively" a differential operator.

Firstly, let us define a central concept in the theory of differential operators.

Definition A.2. Let $P : \Gamma(E_1) \to \Gamma(E_2)$ be a differential operator on M. We say that P has *order* m at $x_0 \in M$ if m is the largest nonnegative integer such that there is a smooth function f vanishing at x_0 such that

$$P(f^m s)(x_0) \neq 0$$

for some section $s \in \Gamma(E_1)$ We thus write $\mathrm{Ord}(P, x_0) = m$. We then define the order of P as

$$\mathrm{Ord}\, P = \sup_{x \in M} \{\mathrm{Ord}(P, x)\}.$$

It is not too hard to check that this definition agrees with the usual definition on $\mathbb{R}^n$ for differential operators over functions. As an example, consider P on $\mathbb{R}$ defined by

$$P(g) = \frac{d^m g}{dx^m}.$$

If $f(x_0) = 0$, it is immediate that $P(f^r)(x_0) = \frac{d^m f^r}{dx^m}(x_0) = 0$ for every $r > m$. On the other hand if $r = m$ let $f(x) = x$ and observe that

$$\frac{d^m x^m}{dx^m} = m! \neq 0.$$

A problem that arises within this definition is that even in the simplest cases, if M is not compact, the order of a linear differential operator may not exist. For example, if $M = \mathbb{R}$, take for each $n \in \mathbb{N}$ a function

$$\phi_n \in C^\infty(\mathbb{R})$$

such that $\mathrm{Supp}(\phi_n) \subseteq [n, n+1]$ and $\phi_n\left(n + \frac{1}{2}\right) > 0$. We then define

$$P(f)(x) = \sum_{n \in \mathbb{N}} \phi_n(x) \frac{d^n f}{dx^n}.$$

It is not hard to see that that for this linear differential operator the order is not defined. In such cases we will say that P is an operator of infinite order.

We will conclude this brief introduction to differential operators by stating the theorem that clarifies why we call such linear operators differential operators. We will first state it on $\mathbb{R}^n$ and then prove it on general Riemannian manifolds. If A, B are two subsets of a topological space X we say that A is compactly contained in B and write $A \Subset B$ if $\overline{A} \subseteq B$ and $\overline{A}$ is compact.

Theorem A.2 (Local Petree theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and let P be a linear differential operator on Ω from the sections of the trivial vector bundle of dimension s $C^\infty(\Omega, s)$ to the sections of the trivial vector bundle of dimension t $C^\infty(\Omega, t)$. Let $\Omega_1 \Subset \Omega$. Then there exists an $m \in \mathbb{N}$ such that for every multi-index α such that $|\alpha| \leq m$ there are maps*

$$a_{\alpha,i,j} \in C^\infty(\Omega_1)$$

such that for any $f \in C^\infty(\Omega_1, s)$

$$(Pf)_j(x) = \sum_{i=1}^{s} \sum_{|\alpha| \leq m} a_{\alpha,i,j}(x)(D^\alpha f)(x) \quad \forall x \in \Omega_1 \ \forall j \leq t.$$

In order to prove such a theorem it is necessary to expand a bit more the theory of differential operators, which, however, is beyond the scope of this appendix (the proof can be found of course in reference [Kah80]). As anticipated, this theorem translates globally on smooth manifolds

Corollary A.3 (Global Petree theorem)**.** *Let M be a smooth manifold and let P be a differential operator on M from an s-dimensional vector bundle to a t-dimensional vector bundle. Let $x_0 \in M$. Then there is a neighbourhood $U \ni x_0$, a chart $\varphi : U \to \Omega \subseteq \mathbb{R}^n$ that trivializes both bundles and a positive integer m such that in the local trivializations*

$$((Pf) \circ \varphi^{-1})_j(x) = \sum_{i=1}^{s} \sum_{|\alpha| \le m} a_{\alpha,i,j}(x)(D^\alpha (f \circ \varphi^{-1})_i)(x) \quad \forall x \in \Omega \ \forall f \in C^\infty(U) \ \forall j \le t.$$

Proof. Let $U \ni x_0$ be a neighbourhood of x_0 such that there is a chart $\varphi : U \to \Omega \in \mathbb{R}^n$. Define $\tilde{P} : g \in C^\infty(\Omega, s) \mapsto (P(g \circ \varphi)) \circ \varphi^{-1}$, which is clearly a differential operator, and apply theorem A.2 to get

$$\left(\tilde{P} g\right)_j (x) = \sum_{i=1}^{s} \sum_{|\alpha| \le m} a_{\alpha,i,j}(x)(D^\alpha g)(x) \quad \forall x \in \Omega_1 \Subset \Omega.$$

By restricting φ we can suppose $\Omega_1 = \Omega$. We therefore get

$$(Pf)_j(x) = \tilde{P}(f \circ \varphi^{-1})_j(\varphi(x)) = \sum_{i=1}^{s} \sum_{|\alpha| \le m} a_{\alpha,i,j}(x)(D^\alpha (f \circ \varphi^{-1})_i)(\varphi(x)) \quad \forall x \in U,$$

which, by precomposing with φ^{-1}, is equivalent to the thesis. □

By locally defining partial derivatives for M as

$$D^\alpha_\varphi f \stackrel{\text{def}}{=} (D^\alpha (f \circ \varphi^{-1})) \circ \varphi,$$

the thesis of theorem A.3 for smooth functions may be restated as

$$(Pf)(x) = \sum_{|\alpha| \le m} a_\alpha(x)(D^\alpha_\varphi f)(x) \quad \forall x \in U,$$

which really looks like the definition of a differential operator in $\mathbb{R}^n$.

This theorem shows that a linear differential operator has a meaningful order locally. It is thus a straightforward corollary that the order is well defined for compact manifolds.

A.2 The divergence and the Laplacian

An easy example of differential operator of order 1 on a generic Riemannian vector bundle is the covariant derivative ∇, since it certainly holds that for every real smooth functions $\operatorname{Supp}(\nabla f) \subseteq \operatorname{Supp}(f)$.

We now turn our attention to a particular differential operator, that is to say the Laplace-Beltrami operator. It is a generalization of Laplace's operator on the Euclidean space, and it is extended to general Riemannian manifolds for example by exploiting the property that Laplace's operator is equal to the divergence of the gradient, even if there are many other possible definitions. Let us therefore generalize these two operators to Riemannian manifolds first.

Divergence Let (M, g) be a Riemannian manifold. For the sake of this essay we might as well suppose the manifold to be orientable, and hence to be endowed with a volume form dV_g. One of the many ways to define the divergence of a vector field $X \in \mathfrak{X}(M)$ is define it as the scalar function $\operatorname{div} X \in C^\infty(M)$ such that

$$(\operatorname{div} X) dV_g = \mathcal{L}_X dV_g,$$

where $\mathcal{L}_X dV_g$ is the Lie derivative of the n-form dV_g. It is well defined because the n-forms space is one dimensional and therefore it can only change by multiplication by a scalar. This definition agrees with our intuition of what the divergence is. Indeed, if we are close to a well of a vector field the flow generated by the field will compress the volume around the well and therefore the Lie derivative of the volume form will be negative. A similar situation holds for sources. Alternatively divergence can be defined through the connection as

$$\operatorname{div} X = \operatorname{Tr}(\nabla X).$$

In both definitions it depends on the metric (indeed that we are dealing with the Levi-Civita connection).

We following property also characterises the divergence:

Lemma A.4. *The divergence is the "formal adjoint" of (the opposite of) both the covariant derivative and the exterior derivative, which on functions are the same thing. This means that for any compactly supported function $f \in C^\infty_C(M)$ and for any vector field $X \in \mathfrak{X}(M)$ we have that*

$$\int_M df(X)\, dV_g = -\int_M f \operatorname{div}(X)\, dV_g.$$

In the above lemma, "formal adjoint" is in quotes because we do not have an inner product between forms and vector fields, but we can still evaluate the form on the field.

Proof. By Stokes theorem we get that

$$\begin{aligned}\int_M (f \operatorname{div}(X) + X(f))\, dV_g &= \int_M (f\mathcal{L}_X + \mathcal{L}_X(f))\, dV_g \\ &= \int_M \mathcal{L}_X (f dV_g) = \int_M \mathrm{d}\left(\iota_X (f dV_g)\right) = \int_{\partial M} \iota_X (f dV_g)\,.\end{aligned}$$

If f has compact support (or if the M has no boundary for non compactly supported functions) the last integral vanishes, so the thesis follows. ☐

The above property is better expressed if we define the divergence on differential 1-forms. As one might expect, if ω is a 1-form on M we define

$$\operatorname{div}(\omega) \overset{\text{def}}{:=} \operatorname{div}(\omega^\sharp).$$

With such definition it is immediate that the formula $\operatorname{div} \omega = \operatorname{Tr}(\nabla\omega)$ is still valid. In this case we can properly say that $\operatorname{div} = -\nabla^*$ on smooth functions. Indeed for any smooth compactly supported function f

$$\langle\!\langle f, \operatorname{div} \omega\rangle\!\rangle = \langle\!\langle f, \operatorname{div} \omega^\sharp\rangle\!\rangle = \int_M f \operatorname{div} \omega^\sharp dV_g = -\int_M df(\omega^\sharp) dV_g = -\int_M \langle df, \omega\rangle dV_g = \langle\!\langle -df, \omega\rangle\!\rangle,$$

where we are denoting with $\langle\!\langle , \rangle\!\rangle$ the L^2 product between functions.

This shows that $-\nabla^*$ is also a good notation for the divergence on 1-forms. The adjoint of d is $-\star d\star$ and hence the divergence of a one form can also be expressed as

$$\operatorname{div} \omega = \star d \star (\omega)$$

and similarly for a vector field X

$$\operatorname{div} X = \star d \star \left(X^\flat\right).$$

The biggest advantage of the expression $\operatorname{div} = -\nabla^*$ however is that it suggests a natural generalization to arbitrary Riemannian vector bundle.

Definition A.3. Let E be a Riemannian vector bundle over M. We define the operator

$$\operatorname{div} = -\nabla^* : \Gamma(T^*M \otimes E) \to \Gamma(E)$$

to be the opposite of the formal adjoint of $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ on compactly supported sections.

Gradient The gradient is basically defined the same way as in the Euclidean space, that is to say, provided that we have a metric, we can use the representation theorem and define the gradient as the vector field that satisfies

$$\langle (\operatorname{grad} f)(x), Y \rangle = df_x(Y) = (\nabla_Y f)(x) = (Y(f))(x) \quad Y \in T_x M.$$

The classic notation ∇ used for the gradient is still a good notation because by the above expression it is clear that the components of the gradient are indeed $\nabla^i f$. The gradient is a differential operator of order 1 as well.

Laplacian Finally, we define the Laplace-Beltrami operator for smooth functions $\Delta : C^\infty(M) \to C^\infty(M)$ as

$$\Delta f \overset{\text{def}}{:=} -\operatorname{div}(\operatorname{grad} f) = \nabla^* \nabla f$$

also known as connection Laplacian because of the last expression. The equivalence of the two expressions is clear from the previous discussion on divergence and gradient. The operator defined this way is positive. However, since when written in standard coordinates in the Euclidean space it turns out to be the opposite of what is usually called the Laplacian, it is often defined with the opposite sign.

As neither the divergence nor the gradient are independent from the metric, we do not expect Laplace operator to be either. Indeed, it turns out that it is dependent on the metric. It is an second order differential operator as can be seen by taking a chart in normal coordinates x^i around a point, by considering the locally defined function x^1 and by using the definition of order.

In coordinates we can see from the definition that

$$\Delta f = -(\det g)^{-1/2} \partial_i \left(g^{ij} (\det g)^{1/2} \partial_j f \right), \tag{A.1}$$

or, more easily, one could write it down in coordinates by using the expression of the divergence as the trace of the covariant derivative

$$\Delta f = -\operatorname{Tr}(\nabla(\nabla f)) = -\operatorname{Tr}(\nabla^2 f) = -g^{ij} \nabla_i \nabla_j f = -g^{ij} f_{;ij}.$$

As for the divergence, the advantage of the expression $\nabla^* \nabla$ is that it is generalizable to arbitrary vector bundles. Indeed, as $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ and $\nabla^* : \Gamma(T^*M \otimes E) \to \Gamma(E)$, it is possible to give the following definition:

Definition A.4. Given a vector bunlde $E \twoheadrightarrow M$, the Laplacian is defined to be the differential operator $\Delta : \Gamma(E) \to \Gamma(E)$ given by $\Delta s = \nabla^* \nabla s$ for all $s \in \Gamma(E)$.

With such expression it is also immediate that the coordinate Laplacian operator is formally self-adjoint.

Proposition A.5. *Let $\varphi, \psi \in \Gamma(E)$ be smooth sections. If one of them is compactly supported we have that*

$$\langle\!\langle \Delta\varphi, \psi \rangle\!\rangle = \langle\!\langle \varphi, \Delta\psi \rangle\!\rangle.$$

Proof.

$$\langle\!\langle \nabla^*\nabla\varphi, \psi \rangle\!\rangle = \langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle = \langle\!\langle \varphi, \nabla^*\nabla\psi \rangle\!\rangle.$$

□

The following lemma is quite handy in calculations.

Lemma A.6. *Let $\{e_i\}$ be a frame and $\{e^i\}$ be its associated orthonormal frame. Let g_{ij} and g^{ij} the associated coordinates. Let $\varphi, \psi \in \Gamma(E)$. Then*

$$\langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle = g^{ij} \langle\!\langle \nabla_i\varphi, \nabla_j\psi \rangle\!\rangle .$$

In particular, if the frame is orthonormal,

$$\|\nabla\varphi\|_{L^2} \leq \sum_i \|\nabla_i\varphi\|_{L^2}$$

where the L^2 norm for Riemannian vector bundles is defined in the next section.

Proof. Note that by definition of the metric on tensor products of Riemannian vector bundles

$$\begin{aligned}\langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle &= \int_M \langle \nabla\varphi, \nabla\psi \rangle dV_g = \int_M \langle \nabla_i\varphi \otimes e^i, \nabla_j\psi \otimes e^j \rangle dV_g \\ &= \int_M \langle e^i, e^j \rangle \langle \nabla_i\varphi, \nabla_j\psi \rangle dV_g = g^{ij} \langle\!\langle \nabla_i\varphi, \nabla_j\psi \rangle\!\rangle .\end{aligned}$$

The second part of the thesis follows by the first part, as

$$\|\nabla\varphi\|_{L^2}^2 = \sum_i \|\nabla_i\varphi\|_{L^2}^2 \leq \left(\sum_i \|\nabla_i\varphi\|_{L^2} \right)^2 .$$

Now it suffices to take square roots. □

The expression $\Delta\varphi = -\operatorname{Tr}(\nabla^2\varphi)$ is still valid for the Laplacian on generic vector bundles, even if in this case the proof is a bit longer (we cannot use the result on the divergence as before because a similar generalization for the divergence makes no sense).

Proposition A.7. *Let $\varphi \in \Gamma(E)$ be a smooth section. Then*

$$\nabla^*\nabla\varphi = -\operatorname{Tr}(\nabla^2\varphi).$$

In particular, on an orthonormal frame

$$\nabla^*\nabla\varphi = -\sum_{j=1}^{n} \nabla^2_{e_j,e_j}\varphi.$$

Proof. Let $\{e_i\}$ be a frame and $\{e^i\}$ be its associated orthonormal frame. Let g_{ij} and g^{ij} the associated coordinates. For all compactly supported ψ it holds

$$\begin{aligned}\langle\!\langle \nabla^*\nabla\varphi, \psi \rangle\!\rangle &= \langle\!\langle \nabla\varphi, \nabla\psi \rangle\!\rangle = g^{ij} \langle\!\langle \nabla_i\varphi, \nabla_j\psi \rangle\!\rangle \\ &= g^{ij} \langle\!\langle -\nabla_i\nabla_j\varphi, \psi \rangle\!\rangle = \left\langle\!\!\left\langle -\nabla_i\nabla^i\varphi, \psi \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle -\operatorname{Tr}(\nabla^2\varphi)\psi \right\rangle\!\!\right\rangle .\end{aligned}$$

To see $g^{ij} \langle\!\langle \nabla_i\varphi, \nabla_j\psi \rangle\!\rangle = g^{ij} \langle\!\langle -\nabla_i\nabla_j\varphi, \psi \rangle\!\rangle$ note that, as the connection is Levi-Civita,

$$g^{ij}\nabla_i\langle \nabla_j\varphi, \psi \rangle = g^{ij}\langle \nabla_i\nabla_j\varphi, \psi \rangle + g^{ij}\langle \nabla_i\varphi, \nabla_j\psi \rangle.$$

Now, as ψ has compact support the term on the left-hand side vanishes because of the divergence theorem when integrated over the whole manifold and the thesis follows. □

Since, as we just saw, differential operators defined through the metric usually depend on the metric used, when we are dealing with more than one metric, we will denote with a subscript the metric with respect to which the differential operator is meant. For example, ∇_g will denote the covariant derivative with respect to the metric g.

A.3 Sobolev spaces on Riemannian manifolds

In this section we define Sobolev spaces on Riemannian manifolds. The main reference for this part is the brief introduction done in [LP87]. In this section M is meant to be a generic Riemannian manifold.

The idea is to adapt the concept of weak derivatives from the euclidean space to Riemannian manifolds. Firstly, we are going to do this for smooth functions and then we will try to generalize the results to more general vector bundles. Let $P : C^\infty(M) \to C^\infty(M)$ be a linear differential operator on M and let $u, f \in L^1_{\text{loc}}(M)$ (that is the set of locally integrable functions on M). We say that $Pu = f$ in a weak sense, or that $Pu \overset{\text{w}}{=} f$, or equivalently that u is a weak solution to the equation $Pu = f$ if it holds

$$\forall \varphi \in C^\infty_C(M) \quad \int_M uP^*\varphi dV_g = \int_M f\varphi dV_g,$$

where $C^\infty_C(M)$ is the set of smooth compactly supported functions on M and P^* is the formal adjoint operator of P on smooth functions.

As it happens for the euclidean space, if it holds $Pu \overset{\text{w}}{=} f$ and $Pu \overset{\text{w}}{=} g$, then $f = g$ almost everywhere on M. Equivalently said, if we consider functions, as it is often done, as classes of equivalence through the relation that identifies two functions equal almost everywhere, the "weak derivative" is unique. This weak definition therefore extends differential operators from the smooth functions to larger subsets of $L^1_{\text{loc}}(M)$ (but not on the whole $L^1_{\text{loc}}(M)$: variations of the counterexamples that work in the Euclidian space work here as well). The spaces we are about to define address the necessity of having spaces closed under weak derivations.

Definition A.5. We define $W^{k,p}(M)$ as the set of all function $u \in L^p(M)$ such that $Pu \in L^p(M)$ (in a weak sense) for all differential operators P of order at most k. We then equip $W^{k,p}(M)$ with the Sobolev norm $\|\cdot\|_{W^{k,p}(M)}$ defined by

$$\|u\|_{W^{k,p}(M)} = \left(\sum_{i=0}^{k} \int_M |\nabla^i u|^p dV_g \right)^{\frac{1}{p}}$$

and we call the resulting Banach space a *Sobolev space* on M.

It is important to note that if M is complete $C^\infty_C(M)$ is dense in $W^{k,p}(M)$.

We can repeat the same construction on vector bundles. Indeed, let E be a Riemannian vector bundle and denote its metric and induced norm by $\langle , \rangle$ and $|\cdot|$. Now given $\varphi, \psi \in \Gamma(E)$ two sections, define

$$\langle\!\langle \varphi, \psi \rangle\!\rangle = \int_M \langle \varphi, \psi \rangle dV_g$$

and denote by $\|\cdot\|$ the norm induced by $\langle\!\langle , \rangle\!\rangle$. This notation highlights the difference between the local product $\langle , \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ and the global product $\langle\!\langle , \rangle\!\rangle : \Gamma(E) \times \Gamma(E) \to \mathbb{R}$ which is a proper inner product on $\Gamma(E)$.

Before defining Sobolev spaces we need to define L^p spaces. Again, we take inspiration from functions. Objects in L^p spaces are very far from smooth functions and we need to implement the same idea in the case of vector bundles. When we write $\Gamma(E)$ we mean smooth sections of the vector bundle E or, if we want to use fancy language, we mean sections in the category of smooth manifolds $\mathbf{Man}^\infty$ of the epimorphism given by the projection π from E onto M. Hence, we write $\Gamma(E) = \Gamma_{\mathbf{Man}^\infty}(E)$. The advantage of using this language is that it is clear what we mean when we write $\Gamma_{\mathbf{Set}}(E)$, that is to say the set of functions which are left inverse of π with no requirements of smoothness or continuity (namely sections of π in the category of sets). For such set the local inner product gives us a generic real function:

$$\langle , \rangle : \Gamma_{\mathbf{Set}}(E) \times \Gamma_{\mathbf{Set}}(E) \to \mathbb{R}^M$$

and hence also the induced norm

$$|\cdot| : \Gamma_{\mathbf{Set}}(E) \to \mathbb{R}^M.$$

The definition of L^p spaces follows naturally

$$L^p(E) = {\{\varphi \in \Gamma_{\mathbf{Set}}(E) \mid |\varphi| \in L^p(M)\}}\big/{\sim}$$

where $\sim$ is the equivalence relation given by $\varphi \sim \psi$ if $|\varphi - \psi| = 0$ almost everywhere. Note that, as in the case of smooth functions, $L^2(E)$ is a Hilbert space with inner product $\langle\!\langle , \rangle\!\rangle$.

Now, given a differential operator between two Riemannian vector bundles $P : \Gamma(E_1) \to \Gamma(E_2)$ it makes sense to talk about its formal adjoint P^* with respect to the product $\langle\!\langle , \rangle\!\rangle$. As for functions, given $\sigma \in L^1_{\text{loc}}(E_1)$ and $\psi \in L^1_{\text{loc}}(E_2)$ we say that $P\sigma = \psi$ in a weak sense, or that $P\sigma \overset{\text{w}}{=} \psi$, or equivalently that σ is a weak solution to the equation $P\sigma = \psi$ if it holds

$$\forall \varphi \in C^\infty_C(E_2) \quad \langle\!\langle \sigma, P^*\varphi \rangle\!\rangle = \langle\!\langle \psi, \varphi \rangle\!\rangle.$$

We can now define Sobolev spaces for generic vector bundles.

Definition A.6. Given E a Riemannian vector bundle over M, we define $W^{k,p}(E)$ as the set of all sections $\sigma \in L^p(E)$ such that $P\sigma \in L^p(M)$ (in a weak sense) for all differential operators $P : \Gamma(E) \to C^\infty(M)$ of order at most k. We then equip $W^{k,p}(E)$ with the Sobolev norm $\|\cdot\|_{W^{k,p}(M)}$ defined by

$$\|\sigma\|_{W^{k,p}(E)} = \left(\sum_{i=0}^{k} \int_M |\nabla^i \sigma|^p dV_g \right)^{\frac{1}{p}}$$

and we call the resulting Banach space a *Sobolev space* on E.

Remark A.1. Note that $\nabla^i u$ is a section of the vector bundle $(T^*M)^{\otimes i} \otimes E$ which is the tensor product of Riemannian bundles and hence it inherits a natural metric. Thus, it makes sense to write $|\nabla^i \sigma|$.

Bibliography

[ADM60a] R. Arnowitt, S. Deser, and C. W. Misner. Canonical variables for general relativity. *Physical Review*, 117(6):1595–1602, mar 1960.

[ADM60b] R. Arnowitt, S. Deser, and C. W. Misner. Energy and the criteria for radiation in general relativity. *Physical Review*, 118(4):1100–1104, may 1960.

[ADM61] R. Arnowitt, S. Deser, and C. W. Misner. Coordinate invariance and energy expressions in general relativity. *Physical Review*, 122(3):997–1006, may 1961.

[Bar86] Robert Bartnik. The mass of an asymptotically flat manifold. *Communications on Pure and Applied Mathematics*, 39(5):661–693, sep 1986.

[DS83] V. I. Denisov and V. O. Soloviev. The energy determined in general relativity on the basis of the traditional hamiltonian approach does not have physical meaning. *Theoretical and Mathematical Physics*, 56(2):832–841, aug 1983.

[Gao04] Chang Jun Gao. Arbitrary-dimensional schwarzschild–FRW black holes. *Classical and Quantum Gravity*, 21(21):4805–4810, oct 2004.

[Kah80] Donald W. Kahn. Chapter 6 differential operators on manifolds. In *Introduction to Global Analysis*, pages 171–205. Elsevier, 1980.

[Kol93] Michor Peter W. Slovak Jan Kolar, Ivan. *Natural Operations in Differential Geometry.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1993.

[Lee19] Dan Lee. *Geometric relativity.* American Mathematical Society, Providence, Rhode Island, 2019.

[LM90] H.Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry (PMS-38), Volume 38.* Princeton University Press, 1990.

[LP87] John M. Lee and Thomas H. Parker. The yamabe problem. *Bulletin of the American Mathematical Society*, 17(1):37–92, jul 1987.

[O'N83] Barrett O'Neill. *Semi-Riemannian Geometry With Applications to Relativity.* Elsevier Science Publishing Co Inc, 1983.

[PT82] Thomas Parker and Clifford Henry Taubes. On witten's proof of the positive energy theorem. *Communications in Mathematical Physics*, 84(2):223–238, jun 1982.

[SY79] Richard Schoen and Shing-Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, feb 1979.

[Wit81] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

[Won] Willie Wai-Yeung Wong. A positive mass theorem for two spatial dimensions.