Remark on connections

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I am writing here such result of a cathartic process I went through in the past couple of hours because I think this consideration is not anywhere else on the web or on famous books.

There are two main ways of defining a connection on a vector bundle $E \twoheadrightarrow M$. One is through the covariant derivative, meant as a map $\nabla : \Gamma(E) \to \Omega^1(M;E) := \Gamma(T^*M \otimes E)$ that satisfies a Leibniz rule. In this case, we can say that covariant derivatives are an affine space over the vector space $\Omega^1(M;\mathrm{End}(E))$. In particular, this means that covariant derivatives are not elements of $\Omega^1(M;\mathrm{End}(E))$ but their differences are.

The second way to look at connections is through matrices of one-forms, with which we can conveniently express a global smooth choice of horizontal subspaces for TE. These subspaces can be written as the intersection of the kernels of $\mathrm{rk}(E)$ one-forms θ^i. Now, let U be an open set of M on which we have both coordinates x^i for M and a trivialization of E. In particular, let V be the standard fiber of E. Choosing a basis for V we have coordinates a^j on V and we can use all this data to have obtain a chart on $\pi^{-1}(E)$ with coordinates x^i, a^j such that the coordinates a^j respect the linear structure of E. With this notation, the matrices of one-forms ω^i_j are such that $\theta^i = da^i + \omega^i_j a^i$.

There is a lot of confusion on many online notes about what these matrices exactly represent. In particular, I have read in multiple locations that they locally define elements of $\Omega^1(M;\mathrm{End}(E))$, or that they make up an affine space over $\Omega^1(M;\mathrm{End}(E))$. I believe neither of these statements is true. Many people think of them just as tables of one-forms that can be written down only in coordinates, but if we want to interpret them as more abstract objects I think the following explanation could help. The correct interpretation of the forms θ^i is that they give the coordinates of a surjective projection on the vertical space, which I denote as V_vE. Indeed, the horizontal subspaces are better thought as the kernel of a projection at each point of TE, that having rank $\dim(V_pE) = \dim(E)$ has the correct dimension of the kernel. Hence the matrices $\omega^i_j a^i$ represent local maps that take a point $p \in M$, a vector $X \in T_pM$, a vector $v \in E_p$ and spit out a vector of V_vE. Because of their codomain, it is clear they have nothing to do with elements of $\Omega^1(M;\mathrm{End}(E))$. There is no any canonical identification whatsoever of V_vE_p and E_p.

This interpretation, together with the remark that a choice of surjective projections on VE at every point of TE makes sense globally and not just in the trivializations, leads to a very beautiful definition of connections that generalizes to any fiber bundle (which is definition 9.3 in the book Natural operations in differential geometry by Kolar, Michor, Slovak). A connection is a global choice of subjective projections on VE, that is

$$\mathrm{Conn}(E) = \{\Phi \in \Omega^1(E;VE) \mid \forall v \in E \ (\Phi_v \circ \Phi_v = \Phi_v) \text{ and } \mathrm{Im}(\Phi_v) = V_vE\}.$$

The requirement $\forall v \in E \ \Phi_v \circ \Phi_v = \Phi_v$ prevents connections from being a vector space: projections are not a vector space. However, projections on a fixed subspace are an affine space. On a vector space, they are an affine space over the vector space of linear transformations sending the kernel of any of the projections to the image. In our case, let $\Phi \in \Omega^1(E;VE)$ be arbitrary and let $H^\Phi E$ be the vector bundle of horizontal subspaces which are the kernel of Φ at every point. Then $\mathrm{Conn}(E)$ is a vector space over $\Gamma(\mathrm{Hom}(H^\Phi E;VE))$. The choice of Φ is arbitrary: $\mathrm{Conn}(E)$ is an affine space over many different vector spaces at the same time, which are evidently all isomorphic. As a vector bundle over M $\mathrm{Hom}(H^\Phi E;VE)$ has rank $\dim(V)^2 \cdot \dim(M)$ which is the same rank of $\Omega^1(E;VE)$ so things make sense.

What I would like to understand now is whether, for each Φ, there is an isomorphism of affine spaces between the two interpretations, which also imply a vector space isomorphism between $\mathrm{Hom}(H^\Phi E;VE)$ and $\Omega^1(M;\mathrm{End}(E))$, without using local trivializations: these matrices of one-forms are really needed? In case, is there any profound reason behind why I need to do things locally?